Uncertainty Relations and Entanglement for PQ-deformed Super Qubit States

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Outlines

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1. Bit, Qubit and Super-Qubit

Bit, Qubit and Super-Qubit

Binary-base 2 representation of numbers -

$a_n, a_{n-1}, ..., a_1, a_0 = 0, 1$ - bits

n-Bit numbers

$$a_n a_{n-1} ... a_1 a_0 = a_n 2^n + a_{n-1} 2^{n-1} + ... a_1 2 + a_0$$

n-Qubit quantum states

$$|a_n\rangle |a_{n-1}\rangle ... |a_1\rangle |a_0\rangle = |a_n 2^n + a_{n-1} 2^{n-1} + ... a_1 2 + a_0\rangle$$

n-Super-Qubit quantum states

$$|a_n\rangle_{ss} |a_{n-1}\rangle_{ss} ... |a_1\rangle_{ss} |a_0\rangle_{ss} = |a_n 2^n + a_{n-1} 2^{n-1} + ... a_1 2 + a_0\rangle_{ss}$$

Qubit as Superposition of States

The unit of quantum information - superposition of two basis states (Bloch sphere - θ, ϕ)

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The number states are eigenstates of fermionic number operator

$$N_f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$N_f|0\rangle = 0|0\rangle, \quad N_f|1\rangle = 1|1\rangle$$

Super-Qubit as Entangled Superposition of States

Superposition of $n = 0$ and $n = 1$ states (super-Bloch sphere θ, ϕ)

$$|\theta, \phi, \zeta\rangle = \cos\frac{\theta}{2}|0, \zeta\rangle + \sin\frac{\theta}{2}e^{i\phi}|1, \zeta\rangle,$$

where the super-number states

$$|0\rangle_{ss} = |0, \zeta\rangle = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}, \quad |1\rangle_{ss} = |1, \zeta\rangle = \frac{1}{\sqrt{1 + |\zeta|^2}} \begin{pmatrix} |1\rangle \\ \zeta|0\rangle \end{pmatrix}$$

(ζ - complex parameter) $\rightarrow$ eigenstates of the super-symmetric number operator

$$\mathcal{N}|0, \zeta\rangle = 0|0, \zeta\rangle, \ \ \mathcal{N}|1, \zeta\rangle = 1|1, \zeta\rangle$$

$|1, \zeta\rangle \rightarrow$ the boson-fermion entangled state

Boson-Fermion Super-Qubit

Realization of qubit $\rightarrow$ arbitrary two level quantum system:

1) spin $s = \frac{1}{2}$: $|\uparrow\rangle, |\downarrow\rangle$

2) single fermion (Pauli principle): $|0\rangle, |1\rangle$

Interaction of two level quantum system with bosons (photons) $\rightarrow$ (Jaynes-Cummings model) is described by the fermion and the boson degrees of freedom.

Question: could we define the unit of quantum information in fermion-boson system?

Here we introduce the analog of qubit in such system, which we call the supersymmetric qubit or shortly, the super-qubit.

For simplicity, we will work with supersymmetric quantum oscillator and with super-quanta or super-particles.

Entangled One Super-particle State

1) Super-number operator $\mathcal{N}$ - counts the total number of super-particles in given state (without distinguishing fermions from bosons)

2) superposition of $|0\rangle_{ss}$ and $|1\rangle_{ss}$ states $\rightarrow$ the super-qubit unit of quantum information, parametrized by two angles on unit sphere (super-Bloch sphere).

3) In contrast to qubit case $\rightarrow$ $|1\rangle_{ss}$ is not unique. It is degenerate state in a superposition of one boson and one fermion states.

4) It requires extra parameters and is characterized by two spheres $S^2 \times S^2$. Information contents of super-qubit state more reach and can be characterized by entanglement of fermionic and bosonic states.

pq-Boson-Fermion Super-qubit States

q- and pq- deformations of bosonic algebra (Quantum Groups)
The pq-deformed super-number operator $[\mathcal{N}]_{p,q}$ and $|0\rangle_{ss,pq}$ and $|1\rangle_{ss,pq}$ eigenstates $\rightarrow$ pq-super-qubit state:

$$|\Psi\rangle = c_0|0\rangle_{ss,pq} + c_1|1\rangle_{ss,pq}$$

1) The states are entangled and for q as a root of the unity, $\rightarrow$ finite number of q-bosons.
2) Superposition of super-coherent states $\rightarrow$ kaleidoscope of finite number of quantum states with quantum group symmetry
It is suitable for description of the supersymmetric qutrit, ququad and in general, supersymmetric qudit unit of quantum information.

2. Supersymmetric Quantum Mechanics

Supersymmetric Quantum Mechanics

1) E. Witten (1981) - SUSY - supersymmetric quantum mechanics

2) Physical realizations - motion of electron in magnetic field, Pauli equation

3) Isospectral quantum systems, integrable models and solitons, Darboux and Bäcklund transformations, Schrödinger factorization, dynamical symmetry

Let a and $a^\dagger$ - bosonic operators, f and $f^\dagger$ - fermionic operators

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I, \quad \{f, f^\dagger\} = ff^\dagger + f^\dagger f = I$$

Hamiltonian of composition

$$H = H_b + H_f = \frac{\hbar\omega}{2}\{a, a^\dagger\} + \frac{\hbar\omega}{2}[f, f^\dagger]$$

Supersymmetric Quantum Oscillator

$$H = \hbar\omega\left(N + \frac{1}{2}\right) + \hbar\omega\left(N_f - \frac{1}{2}\right) = \hbar\omega\mathcal{N}$$

supersymmetric number operator

$$\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & N+1 \end{pmatrix} = N_f \otimes I_b + I_f \otimes N.$$

a^+, a, and the number operator $N = a^\dagger a$, $N + I = aa^\dagger$, are acting in Fock space $\mathcal{H}_b$, with n-particle states $\{|n\rangle\}$ as a basis. The states of supersymmetric quantum oscillator from $\mathcal{H}_f \otimes \mathcal{H}_b$ are characterized by number of fermions $n_f = 0, 1$ and number of bosons $n_b = 0, 1, 2, ...$ The basis states -

$$|0, n\rangle_{ss} = |0\rangle_f \otimes |n\rangle, \qquad |1, n\rangle_{ss} = |1\rangle \otimes |n\rangle$$

Supersymmetric Quantum Oscillator

Then, arbitrary state can be represented by superposition

$$|\Psi\rangle_{ss} = \sum_{n=0}^{\infty} c_{0n}|0\rangle_f \otimes |n\rangle + \sum_{n=0}^{\infty} c_{1n}|1\rangle_f \otimes |n\rangle.$$

The super-number operator, counting total number of super-particles (fermions and bosons) in a state

$$\mathcal{N}|0\rangle_f \otimes |n\rangle = (0+n)|0\rangle_f \otimes |n\rangle$$

$$\mathcal{N}|1\rangle_f \otimes |n\rangle = (1+n)|1\rangle_f \otimes |n\rangle$$

Then, the states $|0\rangle_f \otimes |n\rangle$ and $|1\rangle_f \otimes |n-1\rangle$ contain the same number n of superparticles. Moreover, any superposition of these states also have n superparticles.

Supersymmetric Quantum Oscillator

Spectrum of SUSY quantum oscillator

$$H|n_f, n_b\rangle_{ss} = E_{n_f,n_b}|n_f, n_b\rangle_{ss}$$

is

$$E_{n_f,n_b} = \hbar\omega\left(n_b + \frac{1}{2}\right) + \hbar\omega\left(n_f - \frac{1}{2}\right) = \hbar\omega(n_b + n_f) = \hbar\omega\, n$$

$n = n_b + n_f$ - counts total number of superparticles

Supersymmetric Number States

The normalized generic n super-number state

$$|n,\zeta\rangle_{ss} = \frac{|0,n\rangle_{ss} + \zeta|1,n\rangle_{ss}}{\sqrt{1+|\zeta|^2}} = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} |n\rangle \\ \zeta|n-1\rangle \end{pmatrix}$$

where ζ is an arbitrary complex number, is the eigenstate of super-number operator

$$\mathcal{N}|n,\zeta\rangle = n|n,\zeta\rangle$$

The origin of complex plane $\zeta = 0$ corresponds to n pure bosons, while the infinity in extended complex plane $\zeta = \infty$, to one fermion and $n-1$ bosons.

Stereographic Projection

By stereographic projection, the extended complex plane can be projected to the unit sphere by formula

$$\zeta = \tan\frac{\theta_1}{2}e^{i\phi_1}, \quad (1)$$

so that the state becomes

$$|n,\theta_1,\phi_1\rangle = \cos\frac{\theta_1}{2}\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta_1}{2}e^{i\phi_1}\begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}, \quad (2)$$

where $0 \leq \theta_1 \leq \pi$, $0 \leq \phi_1 \leq 2\pi$ are angles on the sphere.
The states are entangled

Supercharge

$$Q = f^\dagger \otimes a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad Q^\dagger = f \otimes a^\dagger = \begin{pmatrix} 0 & a^\dagger \\ 0 & 0 \end{pmatrix}$$

$(\hbar = 1, \omega = 1)$

- change bosonic state to fermionic one and vice versa
- symmetry operators of H

$$H = \{Q, Q^\dagger\} = \begin{pmatrix} a^\dagger a & 0 \\ 0 & aa^\dagger \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

$$[Q, H] = 0, \qquad [Q^\dagger, H] = 0$$

Supersymmetry and Isospectrality

In coordinate representation

$$\begin{aligned} H_+ &= a^\dagger a = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2} \\ H_- &= a a^\dagger = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{1}{2} \end{aligned}$$

generalization to arbitrary potentials $V_1(x)$, $V_2(x)$,

$$\begin{aligned} H_1 &= A^\dagger A = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x) \\ H_2 &= A A^\dagger = -\frac{1}{2}\frac{d^2}{dx^2} + V_2(x) \end{aligned}$$

Supersymmetry and Isospectrality

Operators

$$A = \frac{1}{\sqrt{2}}\frac{d}{dx} + W(x), \quad A^\dagger = -\frac{1}{\sqrt{2}}\frac{d}{dx} + W(x)$$

$W(x)$ - superpotential

$$V_1(x) = W^2(x) - \frac{1}{\sqrt{2}}W'(x), \quad V_2(x) = W^2(x) + \frac{1}{\sqrt{2}}W'(x)$$

- Riccati equations

$V_1(x)$ and $V_2(x)$ - supersymmetric partner potentials

$$V_2(x) = V_1(x) + \sqrt{2}W'(x)$$

Supersymmetry and Entanglement

The vacuum state $n_f = 0$, $n_b = 0$ - spontaneous supersymmetry breaking state -

$$|\Psi_0\rangle = |0\rangle_f \otimes |0\rangle_b = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}$$

is the fermion-boson separable state

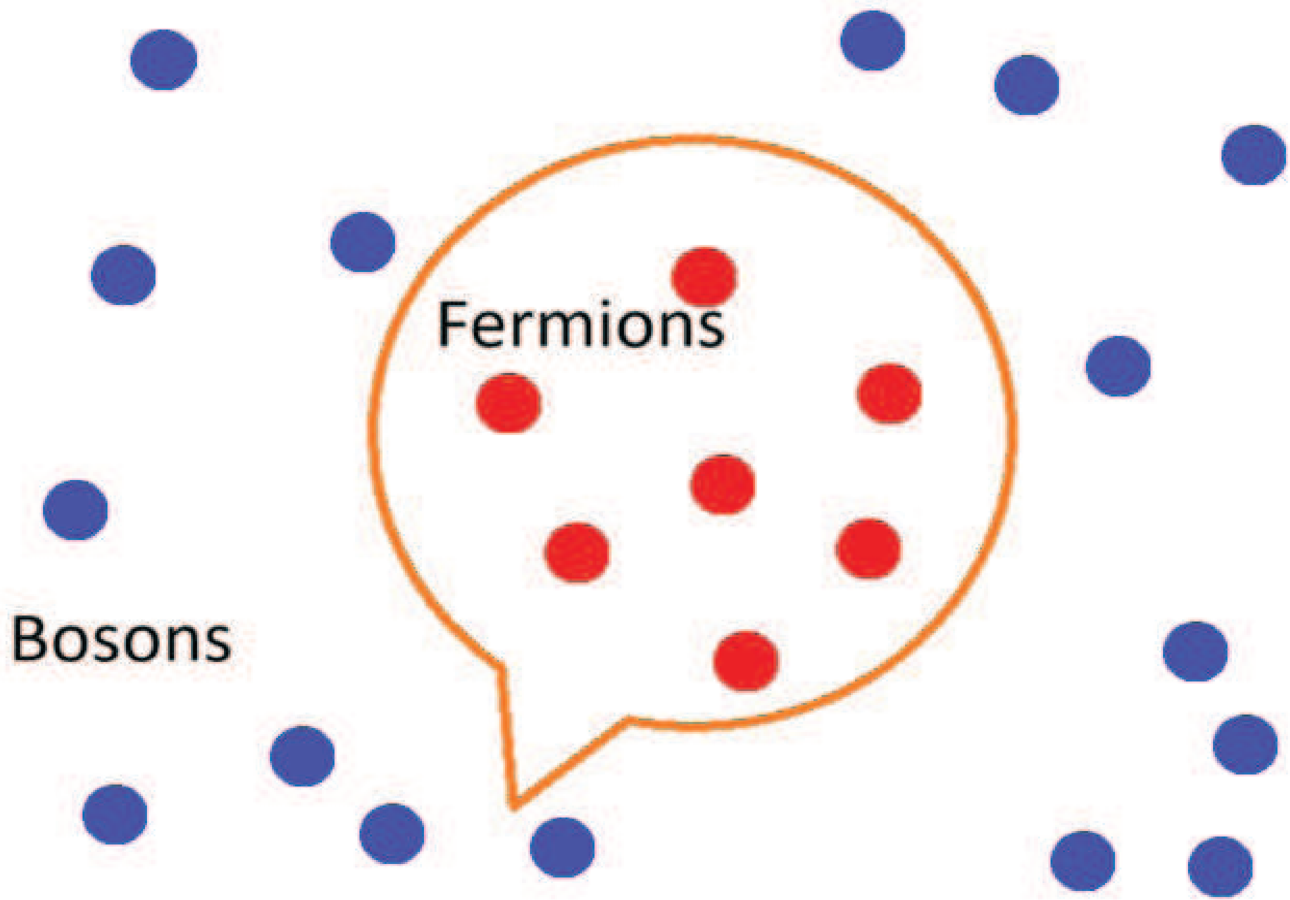
The n-particle state - supersymmetric state - double degenerate

$$|n, \theta, \phi\rangle = \cos\frac{\theta}{2} \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2} e^{i\phi} \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}$$

is fermion-boson entangled state

Superparticles

Boson-Fermion Entanglement Entropy



Fermion-Boson Entanglement

Elementary particles:

Fermions (spin = $\frac{1}{2}$,...; anti-commutator, Fermi-Dirac statistics)

and

Bosons (spin = 1,...; commutator, Bose-Einstein statistics)

Fermions - two states $|0\rangle$ and $|1\rangle$

Bosons - ∞ number of states $|0\rangle$, $|1\rangle$, ..., $|n\rangle$, ...

Fermion-Boson quantum system: Hilbert space $\mathcal{H} = \mathcal{H}_f \otimes \mathcal{H}_b$

- ingredients of any quantum field theory

Supersymmetry - symmetry between fermions and bosons

N=2 Supersymmetric Quantum Mechanics - simplest supersymmetric model - supersymmetric quantum oscillator

Fermion and qubit States

Let f and $f^\dagger$ are fermionic annihilation and creation operators,

$$ff^\dagger + f^\dagger f = \mathbf{I}$$

Eigenstates $|0\rangle_f$ and $|1\rangle_f$ of $N_f = f^\dagger f$, corresponding to fermionic numbers $n_0 = 0$ and $n_1 = 1$

$$N_f|0\rangle_f = 0|0\rangle_f, \quad N_f|1\rangle_f = 1|1\rangle_f$$

qubit basis states. Normalized linear combination → the qubit unit of quantum information

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle_f + \sin\frac{\theta}{2}e^{i\phi}|1\rangle_f, \tag{3}$$

parametrized by points on the Bloch sphere S^2: $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

Boson States

a and $a^\dagger$ are bosonic annihilation and creation operators,

$$aa^\dagger - a^\dagger a = \mathrm{I}$$

Number operator $N = a^\dagger a$ and boson number states (basis in Fock space)

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad a|0\rangle = 0$$

Coherent States: $a|\alpha\rangle = \alpha|\alpha\rangle$, in quantum optics - Glauber (minimization of Heisenberg uncertainty relations - closest to classical states) $\alpha \in \mathcal{C}$ complex number

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = D(\alpha)|0\rangle$$

Qubit-Qudit states

To work with fermionic and bosonic states, we first introduce the qubit-qudit state from Hilbert space $\mathcal{H}_f \otimes \mathcal{H}_n$, and then, to have the Fock space for bosons we take the limit $n \to \infty$. The qudit state is determined by basis vectors $|0\rangle, |1\rangle, ..., |n-1\rangle$ as computational states, and generic qubit-qudit state is

$$|\Psi\rangle = \sum_{k=0}^{n-1} c_{0k}|0\rangle_f \otimes |k\rangle + \sum_{k=0}^{n-1} c_{1k}|1\rangle_f \otimes |k\rangle.$$

The state can be rewritten in two different forms. The first one

$$|\Psi\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle = \left(\begin{array}{c} |\psi_0\rangle \\ |\psi_1\rangle \end{array} \right), \qquad (4)$$

Qubit-Qudit states

represents it in terms of the pair of one qudit states

$$|\psi_0\rangle = \sum_{k=0}^{n-1} c_{0k}|k\rangle, \quad |\psi_1\rangle = \sum_{k=0}^{n-1} c_{1k}|k\rangle. \qquad (5)$$

In the second one,

$$|\Psi\rangle = |\varphi_0\rangle_f \otimes |0\rangle + ... + |\varphi_{n-1}\rangle_f \otimes |n-1\rangle = \sum_{l=0}^{n-1} |\varphi_l\rangle_f \otimes |l\rangle \quad (6)$$

it is given by n, the one qubit states $|\varphi_l\rangle$, $l = 0, ..., n-1$, defined as

$$|\varphi_l\rangle = \begin{pmatrix} c_{0l} \\ c_{1l} \end{pmatrix} = c_{0l}|0\rangle_f + c_{1l}|1\rangle_f. \qquad (7)$$

Now, we send dimension of the qudit state, $n \to \infty$, so that the space of states H_n becomes the Fock space $\mathcal{H}_b$: $\lim_{n\to\infty} \mathcal{H}_n = \mathcal{H}_\infty \equiv \mathcal{H}_b$ and the computational basis of qudit states transforms to Fock number states $|k\rangle_\infty \equiv |k\rangle$, $k = 0, 1, 2, ...$.

3. Entanglement Entropy

Density Matrix for Pure States

for pure state $|\Psi\rangle$:

$$\rho = |\Psi\rangle\langle\Psi|$$

$\rightarrow \rho^2 = |\Psi\rangle\langle\Psi|\Psi\rangle\langle\Psi| = \langle\Psi|\Psi\rangle\rho$

$\rightarrow$

$$\rho^2 = \rho \, tr \rho$$

for pure states $\rightarrow$

$$tr\rho^2 = (tr\rho)^2$$

For normalized states $tr\rho = \langle\Psi|\Psi\rangle = 1$

$\rightarrow$

$$tr\rho^2 = 1$$

Density Matrix for Mixed States

for mixed states

$$\rho = \sum_{i=1}^{k} p_i |\Psi_i\rangle\langle\Psi_i|$$

the difference

$$(tr\,\rho)^2 - tr\,\rho^2 = \sum_{i,j=1}^{k} p_i p_j \langle\Psi_i|\Psi_i\rangle\langle\Psi_j|\Psi_j\rangle - \sum_{i,j=1}^{k} p_i p_j |\langle\Psi_i|\Psi_j\rangle|^2$$

$$= \sum_{i,j=1}^{k} p_i p_j \left(\langle\Psi_i|\Psi_i\rangle\langle\Psi_j|\Psi_j\rangle - |\langle\Psi_i|\Psi_j\rangle|^2\right)$$

Concurrence as Measure of Mixture

$$\frac{1}{2}C^2 = (tr\,\rho)^2 - tr\,\rho^2$$

the Gram determinant form

$$C^2 = 4\sum_{i<j}^{k} p_i p_j \begin{vmatrix} \langle\Psi_i|\Psi_i\rangle & \langle\Psi_i|\Psi_j\rangle \\ \langle\Psi_j|\Psi_i\rangle & \langle\Psi_j|\Psi_j\rangle \end{vmatrix}$$

For normalized states

$$tr\rho = \sum_{i=1}^{k} p_i\langle\Psi_i|\Psi_i\rangle = \sum_{i=1}^{k} p_i = 1$$

$\rightarrow$

$$\frac{1}{2}C^2 = 1 - tr\,\rho^2$$

Frobenius Norm and Linear Entropy for Mixed States

Then the level of mixture in the state with Hermitian density matrix $\rho = \rho^\dagger$ is determined by the Frobenius norm

$$tr\rho^2 = tr\rho^\dagger\rho = \sum_{i,j=1}^{k} |(\rho)_{ij}|^2 \equiv ||\rho||_F^2$$

and its deviation from the unity $\rightarrow$
the linear entropy S_L and the concurrence C expressed by this norm

$$S_L = 1 - tr\rho^2 = 1 - ||\rho||^2 = \frac{1}{2}C^2$$

so that the positive number

$$0 \leq C \leq \sqrt{2}$$

Frobenius Sphere for Pure States

For pure states $tr\,\rho^2 = 1 \rightarrow C = 0$ and

$$||\rho||_F^2 = 1$$

This equation describes the unit radius sphere, which we call as the Frobenius sphere. Every point on this sphere corresponds to a pure state.

Frobenius Spherical Shell for Mixed States

If

$$tr\,\rho^2 < 1$$

the concurrence $C \neq 0$ and the state is mixed.

The level of mixture can be characterized by the concurrence C.

By using the Frobenius norm

$$||\rho||_F^2 = \sum_{i=1}^{k} (\rho)_{ii}^2 + 2 \sum_{1=i<j}^{k} |(\rho)_{ij}|^2$$

and the squared normalization condition $(tr\,\rho)^2 = 1 \rightarrow$

$$\begin{aligned} C^2 &= 2(1 - ||\rho||^2) \\ &= 2(\sum_{i=1}^{k} (\rho)_{ii} \sum_{j=1}^{k} (\rho)_{jj} - \sum_{i=1}^{k} (\rho)_{ii}^2 - 2 \sum_{1=i<j}^{k} |(\rho)_{ij}|^2) \end{aligned}$$

Gram Determinant Form

$$C^2 = 4 \sum_{1=i<j}^{k} \begin{vmatrix} (\rho)_{ii} & (\rho)_{ij} \\ (\rho)_{ji} & (\rho)_{jj} \end{vmatrix}$$

In this formula the sum is going according to all 2×2 minors of the density matrix ρ.

Maximally Mixed States

To find maximal value of concurrence for mixed states, we diagonalize Hermitian matrix ρ by unitary matrix U

$$\rho = U\,\Lambda\,U^{\dagger}, \quad (8)$$

- diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_k)$, and all eigenvalues are real. The unit trace condition becomes

$$\sum_{i=1}^{k} \lambda_i = 1 \quad (9)$$

and for the Frobenius norm we have

$$||\Lambda||_F^2 = \sum_{i=1}^{k} \lambda_i^2 . \quad (10)$$

Maximally Mixed States

This gives the concurrence square as

$$C^2 = 2(1 - \sum_{i=1}^{k} \lambda_i^2) = 4 \sum_{1=i<j}^{k} \lambda_i \lambda_j \qquad (11)$$

To evaluate maximal value of concurrence we consider function

$$F(\lambda_1, ..., \lambda_k, \lambda) = C^2 - \lambda((tr\,\Lambda)^2 - 1)$$

$$= 2(1 - \sum_{i=1}^{k} \lambda_i^2) - \lambda((\sum_{i=1}^{k} \lambda_i)^2 - 1)$$

with Lagrange multiplier λ.

Maximally Mixed States

For critical points → system of equations ($i = 1, 2, ..., k$)

$$\frac{\partial F}{\partial \lambda_i} = -4\lambda_i - 2\lambda \sum_{i=1}^{k} \lambda_i = 0$$

with solution $\lambda_1 = \lambda_2 = ... = \lambda_k = \frac{1}{k}$ Then, for maximally mixed states

$$||\Lambda||_F^2 = \frac{1}{k}$$

Frobenius spherical shell:

$$\frac{1}{k} \leq ||\rho||_F^2 \leq 1$$

Entanglement Entropy

for pure state $\rho = |\Psi\rangle\langle\Psi|$

Bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Reduced density matrix

$$\rho_A = Tr_B \rho$$

von Neumann entropy = entanglement entropy

$$S_A = -Tr(\rho_A \log \rho_A)$$

If $Tr\rho_A^2 = 1$, the reduced state is pure state and the original state $|\Psi\rangle$ is separable:

$$|\Psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \rightarrow \rho_A = |\psi\rangle_{AA}\langle\psi|$$

If $Tr\rho_A^2 < 1$, the reduced state is mixed state and the original state $|\Psi\rangle$ is entangled

EPR pair

Bell state

$$|\Psi\rangle = \frac{|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B}{\sqrt{2}}$$

density matrix $\rho = |\Psi\rangle\langle\Psi|$ $\rightarrow$ reduced density matrix - mixed state

$$\rho_A = Tr_B\rho = \frac{1}{2}(|0\rangle_{AA}\langle 0| + |1\rangle_{AA}\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_A^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow Tr\rho_A^2 = \frac{1}{2}$$

$$\rho_A^n = \frac{1}{2^n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow Tr\rho_A^n = \frac{1}{2^{n-1}}$$

Maximal Entanglement of EPR pair

1) concurrence - the linear entropy

$$C^2 = 2(1 - Tr\rho_A^2) = 1 \;\rightarrow\; C = 1$$

2) von Neumann entropy

$$S_A = -Tr(\rho_A \log \rho_A) = 1$$

3) the Renyi entropy of order n

$$S_A^{(n)} = \frac{1}{1-n} \log_2 Tr\rho_A^n = 1$$

Maximally entangled two qubit state

Entanglement and Entropy of the States

The direct product state of the form $|\psi\rangle_f \otimes |\phi\rangle_b$ is the fermion-boson separable state from the Hilbert space $\mathcal{H}_f \otimes \mathcal{H}_b$. If a state is not separable, then it is entangled.

The fermion-boson basis states are formed by tensor product of the fermionic (the qubit) states with Fock states, $|0\rangle_f \otimes |k\rangle$, and $|1\rangle_f \otimes |k\rangle$, $k = 0, 1, 2, ...$ Then, for a generic state

$$|\Psi\rangle = \sum_{k=0}^{\infty} c_{0k}|0\rangle_f \otimes |k\rangle + \sum_{k=0}^{\infty} c_{1k}|1\rangle_f \otimes |k\rangle, \qquad (12)$$

from $\mathcal{H}_f \otimes \mathcal{H}_b$ Hilbert space we have two representations.

Double Bosonic representation of the states

The first one

$$|\Psi\rangle = |0\rangle_f \otimes |\psi_0\rangle + |1\rangle_f \otimes |\psi_1\rangle = \begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \end{pmatrix}$$

is determined by two bosonic states

$$|\psi_0\rangle = \sum_{k=0}^{\infty} c_{0k}|k\rangle, \quad |\psi_1\rangle = \sum_{k=0}^{\infty} c_{1k}|k\rangle$$

as vectors in the Fock space.

Density matrix

The state $|\Psi\rangle$ is separable if and only if in first representation, two Fock states are linearly dependent, $|\psi_0\rangle = \lambda|\psi_1\rangle$.
If these states are linearly independent, the state is entangled. For normalized state the density matrix is

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} |\psi_0\rangle\langle\psi_0| & |\psi_0\rangle\langle\psi_1| \\ \\ |\psi_1\rangle\langle\psi_0| & |\psi_1\rangle\langle\psi_1| \end{pmatrix}$$

By taking the partial trace in fermionic states, the reduced bosonic density matrix

$$\rho_b = tr_f\, \rho = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$$

Bosonic Gram determinant

We find the concurrence square as determinant of the Hermitian inner product metric $g_{ij} = \langle\psi_i|\psi_j\rangle$, (the Gram determinant), of two vectors $(i, j = 0, 1)$ in Fock space,

$$C^2 = 4 \begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix}$$

the concurrence square can be rewritten as an infinite sum of modulus squares of all 2×2 minors of the coefficient matrix c_{nm},

$$C^2 = 4 \sum_{0=n<m}^{\infty} \left\| \begin{matrix} c_{0n} & c_{0m} \\ \\ c_{1n} & c_{1m} \end{matrix} \right\|^2$$

Concurrence and Mean Value of Spin

spin $s = \frac{1}{2}$ density matrix

$$\hat{\rho} = \frac{1}{2} \begin{pmatrix} 1 + \sigma_3 & 2\sigma_+ \\ 2\sigma_- & 1 - \sigma_3 \end{pmatrix}$$

$$C^2 = 2\, det \begin{vmatrix} 1 + \langle \sigma_3 \rangle & 2\langle \sigma_+ \rangle \\ 2\langle \sigma_- \rangle & 1 - \langle \sigma_3 \rangle \end{vmatrix}$$

$$C^2 = 1 - \langle \sigma \rangle^2$$

$$\langle \sigma_1 \rangle^2 + \langle \sigma_2 \rangle^2 + \langle \sigma_3 \rangle^2 + C^2 = 1$$

$C = 1$ max entangled state $\rightarrow \langle \sigma_i \rangle^2 = 0,\, i = 1, 2, 3$

Frobenius Spherical Shell for Entangled Boson-Fermion States

separable and entangled states admit simple geometrical interpretation by points in spherical shell.
An arbitrary qubit-n-qudit state from Hilbert space $\mathcal{H}_f \otimes \mathcal{H}_n$ with density matrix ρ, by taking partial trace we get reduced density matrix $\rho_n = tr_f \rho \rightarrow$ the linear entropy S_L and the concurrence C expressed by this norm

$$S_L = 1 - tr\rho_n^2 = 1 - ||\rho_n||^2 = \frac{1}{2}C^2, \tag{13}$$

so that

$$C^2 = 2(1 - ||\rho_n||^2). \tag{14}$$

For separable states $tr\rho_n^2 = 1$, $C = 0$, and

$$||\rho_n||^2 = 1. \tag{15}$$

Frobenius Spherical Shell for Entangled States

so that every point on Frobenius sphere corresponds to a separable state from $\mathcal{H}_f \otimes \mathcal{H}_n$. If $tr\rho_n^2 < 1$, the concurrence $C \neq 0$ and the original pure state is entangled. The level of entanglement can be characterized by the concurrence C.

for maximally entangled states

$$||\rho||_F^2 = \frac{1}{n} \tag{16}$$

and the maximal value of concurrence is

$$C_{max}^2 = 2\left(1 - \frac{1}{n}\right) = 2\,\frac{n-1}{n} \tag{17}$$

Frobenius ball and Bosonic states

In particular cases we have:

1) for $n = 2$, the qubit-qubit state, and $C_{max} = 1$

2) for $n = 3$, the qubit-qutrit state, and $C_{\max} = 2/\sqrt{3}$

3) for $n = 4$, the qubit-ququad state, and $C_{max} = \sqrt{3}/\sqrt{2}$

To find the concurrence for maximally entangled fermion-boson states we have to take the limit $n \to \infty$, which gives

$$C_{\max} = \sqrt{2}. \tag{18}$$

The above calculations allow us to find geometrical image of entangled states. In fact, the maximal concurrence implies the Frobenius sphere

$$||\Lambda||_F^2 = \frac{1}{n} \tag{19}$$

with radius $r = 1/\sqrt{n}$.

Frobenius Ball

As a result, we have geometrical classification of entangled states.

1. Qubit-n-Qudit states:

a) The separable qubit-n-qudit states are determined by Frobenius sphere with the radius one, $||\rho_n||^2 = 1$.

b) The entangled states belong to the Frobenius spherical shell with radius $1/\sqrt{n} \leq r < 1$.

2. Fermion-Boson state, when $n \to \infty$:

radius of internal sphere $r = 1/\sqrt{n}$ (maximally entangled states), vanishes $r = 0$ $\to$ only one state at the origin $\to$ entangled fermion-boson states belong to Frobenius ball $||\rho_\infty||^2 < 1$, with maximal value of the concurrence $C = \sqrt{2}$, corresponding to maximally entangled state at the center of the ball.

Frobenius shell and Frobenius ball

Frobenius Shell: $\frac{1}{\sqrt{n}} \leq r_n \leq 1$ Frobenius Ball: $0 \leq r_\infty \leq 1$

$$\sqrt{2\frac{n-1}{n}} \geq C_n \geq 0 \qquad \sqrt{2} \geq C_\infty \geq 0$$

Super Number states

Arbitrary superposition of these two states is also state with n super-quanta - the super-number state

$$|n,\theta,\phi\rangle = \cos\frac{\theta}{2}\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}$$

$\Rightarrow$ energy levels with n super-quanta $E_{n>0} = n\omega$ are double degenerate with arbitrary $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. For $n = 0$ the state $|\Psi_0\rangle = |0\rangle_f \otimes |0\rangle_b$ is the ground state with $E_0 = 0$. The super-number state contains n super-quanta,

$$\mathcal{N}|n,\theta,\phi\rangle = n|n,\theta,\phi\rangle$$

in superposition of the zero fermionic state $|0\rangle_f \otimes |n\rangle$ and the one fermionic state $|1\rangle_f \otimes |n-1\rangle$.

Entanglement of Super-Number States

Then the reduced bosonic, as well as fermionic state is mixed state and the generic state $|n, \theta, \phi\rangle$ is entangled with concurrence

$$C = \sin\theta. \tag{20}$$

It is bounded $0 \leq C \leq 1$ and does not dependent on n. The states along the equator are maximally entangled states

$$|n, \frac{\pi}{2}, \phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |n\rangle + e^{i\phi}|1\rangle \otimes |n-1\rangle). \tag{21}$$

4. Entangled Super-Coherent States

Fermion-Boson Bell States

For $n = 1$ we have the maximally entangled states - fermion-boson analog of Bell states

$$|L_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1\rangle_b \pm |1\rangle_f|0\rangle_b).$$

and another pair of fermionic-bosonic Bell states

$$|B_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|0\rangle_b \pm |1\rangle_f|1\rangle_b).$$

The Bell based Super-qubit States

For every Bell state we have its own annihilation operator, which in addition to bosonic annihilation operator a includes the fermionic annihilation or creation operators, f and $f^\dagger$

$$A_{\pm 1} = \begin{pmatrix} a & \pm 1 \\ 0 & a \end{pmatrix} = I_f \otimes a \pm f \otimes I_b,$$

$$A^T_{\pm 1} = \begin{pmatrix} a & 0 \\ \pm 1 & a \end{pmatrix} = I_f \otimes a \pm f^\dagger \otimes I_b,$$

The Bell based Super-qubit States

The vacuum state is annihilated by two operators

$$A_{\pm 1}|\Psi_0\rangle = 0,$$

and it is orthogonal to the pair of Bell states $|L_+\rangle$ and $L_-\rangle$. By taking superposition of the state with these Bell states we get two normalized reference states,

$$|0, C, \phi\rangle_{L_\pm} = \sqrt{1-C}|\Psi_0\rangle + \sqrt{C}e^{i\phi}|L_\pm\rangle,$$

which are annihilated by operators

$$A_{\mp 1}|0, C, \phi\rangle_{L_\pm} = 0.$$

The states are parametrized by real number C, bounded between $0 \leq C \leq 1$. It represents the concurrence, showing the level of fermion-boson entanglement in the reference state.

Four Bell based reference states

As a result, we have constructed four, the Bell type reference states

$$
\begin{aligned}
|0, C, \phi\rangle_{L_\pm} &= \sqrt{1-C}|\Psi_0\rangle + \sqrt{C}e^{i\phi}|L_\pm\rangle, \\
|0, C, \phi\rangle_{B_\pm} &= \sqrt{1-C}|\Psi_1\rangle + \sqrt{C}e^{i\phi}|B_\pm\rangle,
\end{aligned}
$$

with the inner products

$$
{}_{L_+}\langle 0, C, \phi|0, C, \phi\rangle_{L_-} = 1 - C
$$

$$
{}_{B_+}\langle 0, C, \phi|0, C, \phi\rangle_{B_-} = 1 - C
$$

and corresponding fidelity $F = (1-C)^2$, expressed in terms of the concurrence C.

Bell type Super-qubit state

The reference states

$$|0,\theta,\phi\rangle_{L_\pm} \quad = \quad \cos\frac{\theta}{2}|\Psi_0\rangle + \sin\frac{\theta}{2}e^{i\phi}|L_\pm\rangle$$

- the Bell type super-qubit states.

north pole $\rightarrow$ separable vacuum state $|\Psi_0\rangle \equiv |0\rangle_S$

south pole $\rightarrow$ maximally entangled Bell state $|L_\pm\rangle \equiv |1\rangle_S$

superposition of $n = 0$ and $n = 1$ superparticle states

$$\mathcal{N}|0\rangle_S = 0\,|0\rangle_S \quad \mathcal{N}|1\rangle_S = 1\,|1\rangle_S$$

the state is fermion-boson entangled and the computational basis for this super-qubit state is made from $|0\rangle_S$ and $|1\rangle_S$ eigenstates of super-number operator $\mathcal{N}$.

The Bell Supersymmetric Coherent States

To create entangled super-coherent state we have to choose reference state with entangled bosons and fermions.
We consider set of maximally entangled four Bell reference states as super-qubit states.
The Bell super-coherent states defined as

$$|\alpha, L_\pm\rangle \equiv \mathcal{D}(\alpha)|L_\pm\rangle, \qquad |\alpha, B_\pm\rangle \equiv \mathcal{D}(\alpha)|B_\pm\rangle$$

are eigenstates of corresponding supersymmetric annihilation operators

$$\begin{aligned} A_1|\alpha, L_-\rangle &= \alpha|\alpha, L_-\rangle, \qquad A_{-1}|\alpha, L_+\rangle = \alpha|\alpha, L_+\rangle, \\ A_1^T|\alpha, B_-\rangle &= \alpha|\alpha, B_-\rangle, \qquad A_{-1}^T|\alpha, B_+\rangle = \alpha|\alpha, B_+\rangle. \end{aligned}$$

The Bell Supersymmetric Coherent States

The states are orthonormal and maximally entangled.

In explicit form the states are expressed as

$$|\alpha, L_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|1,\alpha\rangle \pm |1\rangle_f|0,\alpha\rangle),$$

$$|\alpha, B_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_f|0,\alpha\rangle \pm |1\rangle_f|1,\alpha\rangle),$$

in terms of the displaced Fock states

$$|0,\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle,$$

$$|1,\alpha\rangle = D(\alpha)|1\rangle = e^{-\frac{1}{2}|\alpha|^2}(\frac{d}{d\alpha}|\alpha\rangle - \bar{\alpha}|\alpha\rangle)$$

Here $|\alpha\rangle$ is the Glauber coherent state (not normalized).

Bell based Supersymmetric Coherent States

The linear combination of these, maximally entangled states with orthogonal separable states produces the set of four supercoherent states. These states are created by displacement operator, acting on super-qubit reference states

$$|\alpha, C, \phi\rangle_{L_\pm} \equiv \mathcal{D}(\alpha)|0, C, \phi\rangle_{L_\pm}$$

$$|\alpha, C, \phi\rangle_{B_\pm} \equiv \mathcal{D}(\alpha)|0, C, \phi\rangle_{B_\pm}$$

- eigenstates of super annihilation operators

$$A_1|\alpha, C, \phi\rangle_{L_-} = \alpha|\alpha, C, \phi\rangle_{L_-}, \quad A_{-1}|\alpha, C, \phi\rangle_{L_+} = \alpha|\alpha, C, \phi\rangle_{L_+}$$

$$A_1^T|\alpha, C, \phi\rangle_{B_-} = \alpha|\alpha, C, \phi\rangle_{B_-}, \quad A_{-1}^T|\alpha, C, \phi\rangle_{B_+} = \alpha|\alpha, C, \phi\rangle_{B_+}$$

The Supersymmetric Bell based Coherent States

Then we have following definition. The super-coherent states as displaced super-qubit states

$$|\alpha, C, \phi\rangle_{L_-} = \sqrt{1-C}|0\rangle_f \otimes |0, \alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, L_-\rangle,$$
$$|\alpha, C, \phi\rangle_{L_+} = \sqrt{1-C}|0\rangle_f \otimes |0, \alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, L_+\rangle,$$
$$|\alpha, C, \phi\rangle_{B_-} = \sqrt{1-C}|1\rangle_f \otimes |0, \alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, B_-\rangle,$$
$$|\alpha, C, \phi\rangle_{B_+} = \sqrt{1-C}|1\rangle_f \otimes |0, \alpha\rangle + \sqrt{C}e^{i\phi}|\alpha, B_+\rangle.$$

we call as the super-Bell based states.

The Supersymmetric Bell based Coherent States

On the super-Bloch sphere these states take form

$$|\alpha,\theta,\phi\rangle_{L_{\mp}} = \cos\frac{\theta}{2}\begin{pmatrix} |0,\alpha\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix} |1,\alpha\rangle \\ \mp|0,\alpha\rangle \end{pmatrix}$$

$$|\alpha,\theta,\phi\rangle_{B_{\mp}} = \cos\frac{\theta}{2}\begin{pmatrix} 0 \\ |0,\alpha\rangle \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix} |0,\alpha\rangle \\ \mp|1,\alpha\rangle \end{pmatrix}$$

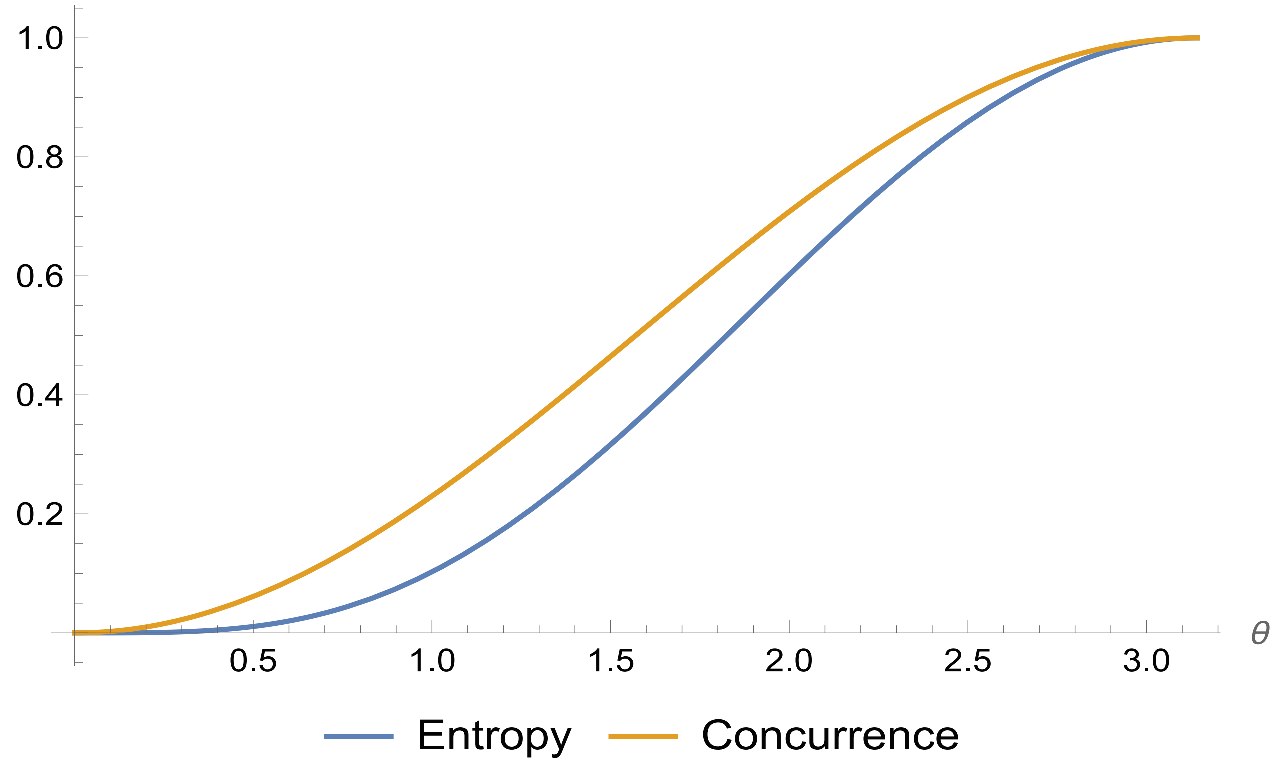
Entanglement of Supercoherent States

For supersymmetric coherent states $|\alpha, C, \phi\rangle_{L_\mp}$, $|\alpha, C, \phi\rangle_{B_\mp}$, the concurrence is independent of α and is equal

$$C = \sin^2 \frac{\theta}{2}. \qquad (22)$$

Concurrence and Entanglement versus θ

Entanglement on super-Bloch sphere



5. Uncertainty Relations and Entanglement for Bell based Super-Coherent States

Uncertainty Relations and Entanglement on Super-Bloch Sphere

Here we calculate the uncertainty relations for quartet of supercoherent states $|\alpha, \theta, \phi\rangle_{L_\pm}$ $|\alpha, \theta, \phi\rangle_{B_\pm}$. The coordinate and momentum operators we define as

$$X = I_f \otimes \frac{1}{\sqrt{2}}(a + a^\dagger)$$

$$P = I_f \otimes \frac{i}{\sqrt{2}}(a^\dagger - a)$$

transformed by displacement operator to

$$\mathcal{D}^\dagger(\alpha) X \mathcal{D}(\alpha) = X + I_f \otimes \frac{\alpha + \bar{\alpha}}{\sqrt{2}} = X + I_f \otimes \sqrt{2}\Re\alpha,$$

$$\mathcal{D}^\dagger(\alpha) P \mathcal{D}(\alpha) = X + I_f \otimes i\frac{\bar{\alpha} - \alpha}{\sqrt{2}} = P + I_f \otimes \sqrt{2}\Im\alpha.$$

Uncertainty Relations and Entanglement

Theorem

Dispersions of coordinate X and momentum P in all super-coherent states $|\alpha, C, \phi\rangle_{L_\pm}$ and $|\alpha, C, \phi\rangle_{B_\pm}$ are the same and equal

$$(\Delta X)^2_\alpha \equiv \langle X^2\rangle_\alpha - \langle X\rangle^2_\alpha = \frac{1}{2}(1+C) - C(1-C)\cos^2\phi$$

$$(\Delta P)^2_\alpha \equiv \langle P^2\rangle_\alpha - \langle P\rangle^2_\alpha = \frac{1}{2}(1+C) - C(1-C)\sin^2\phi$$

They do not depend of α:

$$(\Delta X)^2_\alpha = (\Delta X)^2_0$$

$$(\Delta P)^2_\alpha = (\Delta P)^2_0$$

Uncertainty Relations and Entanglement

The uncertainty relation for the supersymmetric coherent states are found as monotonically growing function of C,

$$\Delta X \Delta P = \frac{1}{2}\sqrt{1 + C^2 + 2C^3 + C^2(1 - C)^2 \sin^2 2\phi},$$

with small periodic dependence on angle ϕ.

This implies inequality

$$\frac{1}{2}\sqrt{1 + C^2 + 2C^3} \leq \Delta X \Delta P \leq \frac{1}{2}(1 + C^2),$$

with minimum value at $\phi = 0$ and the maximal one at $\phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Uncertainty Relations and Entanglement

$$\frac{1}{2} \leq (\Delta X)\,(\Delta P) \leq 1.$$

The lower limit, $C = 0$ and state $|\alpha, \Psi_0\rangle$,

$$(\Delta X)\,(\Delta P) = \frac{1}{2}$$

while the upper limit for $C = 1$, to state $|\alpha, L_\pm\rangle$,

$$(\Delta X)\,(\Delta P) = 1,$$

For zero fermionic state uncertainty reaches minimal value, as pure bosonic coherent state, most classical state and separable one $C = 0$. Mixing bosonic and fermionic degrees $(C \neq 0)$, due to nonclassical nature of fermions, increases non-classicality of the states and corresponding uncertainty.

6. Uncertainty Relations for Super-Coherent States

Qubit and Schrödinger cat states

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$N|0\rangle = 0|0\rangle$$

-zero superparticle state

$$N|1\rangle = 1|1\rangle$$

-one superparticle state

Displaced qubit state

$$D(\alpha)|\Psi\rangle = \cos\frac{\theta}{2}|0,\alpha\rangle + \sin\frac{\theta}{2}e^{i\phi}|1,\alpha\rangle$$

the Schrödinger cat states - macroscopic and orthogonal

Super-Qubit State (Entangled)

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle_{SUSY} + \sin\frac{\theta}{2}e^{i\phi}|1\rangle_{SUSY}$$

$$\mathcal{N}|0\rangle_{SUSY} = 0|0\rangle_{SUSY}$$

-zero particle state

$$\mathcal{N}|1\rangle_{SUSY} = 1|1\rangle_{SUSY}$$

-one particle state

Example:

$$|0\rangle_{SUSY} = |0\rangle_f \otimes |0\rangle_b$$

$$|1\rangle_{SUSY} = |L_\pm\rangle$$

- boson-fermion Bell state

One Super-particle State

general one super-particle state $\mathcal{N}|1,\zeta\rangle = 1|1,\zeta\rangle$,

$$|1,\zeta\rangle = \frac{|0\rangle_f \otimes |1\rangle_b + \zeta|1\rangle_f \otimes |0\rangle_b}{\sqrt{1+|\zeta|^2}} = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} |1\rangle \\ \zeta|0\rangle \end{pmatrix}$$

- fermion-boson entangled one super-particle state.

Stereographic projection of the extended complex plane can be projected to the unit sphere

$$\zeta = \tan\frac{\theta_1}{2} e^{i\phi_1},$$

Concurrence-not maximally entangled

$$C_\zeta = \frac{2|\zeta|}{1+|\zeta|^2}$$

Entanglement of Super-Qubit State

The super-qubit quantum state

$$|\theta, \phi, \zeta\rangle = \cos\frac{\theta}{2}|0, \zeta\rangle + \sin\frac{\theta}{2}e^{i\phi}|1, \zeta\rangle,$$

or in explicit form

$$|\theta, \phi, \zeta\rangle = \cos\frac{\theta}{2}\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} |1\rangle \\ \zeta|0\rangle \end{pmatrix}$$

is characterized by two real θ, ϕ and one complex ζ parameters.

Concurrence

$$C = \sin^2\frac{\theta}{2}\frac{2|\zeta|}{1+|\zeta|^2}$$

Uncertainty Relations for Super-Coherent States

The coordinate and momentum operators defined by

$$X = I_f \otimes \frac{a + a^\dagger}{\sqrt{2}}, \qquad P = I_f \otimes i\frac{a^\dagger - a}{\sqrt{2}},$$

transform by the dislpacement operator as

$$\mathcal{D}^\dagger(\alpha) X \mathcal{D}(\alpha) = X + \sqrt{2}\Re\alpha, \quad \mathcal{D}^\dagger(\alpha) P \mathcal{D}(\alpha) = P + \sqrt{2}\Im\alpha.$$

The average values of X and P operators in super-coherent states are equal

$$\begin{aligned} \langle \alpha, \theta, \phi, \zeta | X | \alpha, \theta, \phi, \zeta \rangle &= \sqrt{2}\Re\alpha + \frac{\sin\theta\cos\phi}{2\sqrt{1 + |\zeta|^2}}, \\ \langle \alpha, \theta, \phi, \zeta | P | \alpha, \theta, \phi, \zeta \rangle &= \sqrt{2}\Im\alpha + \frac{\sin\theta\sin\phi}{2\sqrt{1 + |\zeta|^2}}. \end{aligned}$$

Uncertainty Relations

The dispersions or the variance of X and P operators are defined as $(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$, $(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$. Dispersions of X and P operators in super-coherent states are not dependent on α and equal

$$
\begin{aligned}
(\Delta X)^2 &= \frac{1}{2}\left(1 + \frac{2\sin^2\frac{\theta}{2} - \sin^2\theta\cos^2\phi}{1+|\zeta|^2}\right), \\
(\Delta P)^2 &= \frac{1}{2}\left(1 + \frac{2\sin^2\frac{\theta}{2} - \sin^2\theta\sin^2\phi}{1+|\zeta|^2}\right),
\end{aligned}
$$

Uncertainty Relations

or in terms of the Cartesian coordinates of the super-Bloch sphere

$$(\Delta X)^2 = \frac{1}{2}\left(1 + \frac{1 - z - x^2}{1 + |\zeta|^2}\right),$$
$$(\Delta P)^2 = \frac{1}{2}\left(1 + \frac{1 - z - y^2}{1 + |\zeta|^2}\right).$$

For angle $\phi = \frac{\pi}{4}$, coordinates $x = y$ and dispersions are equal. For the corresponding state in equatorial plane $\theta = \frac{\pi}{2}$,

$$(\Delta X)^2 = (\Delta P)^2 = \frac{1}{2}\left(1 + \frac{1}{2(1 + |\zeta|^2)}\right).$$

Uncertainty Relations and Fibonacci Sequence

From this formula for states with $|\zeta| = 1$ we have

$$(\Delta X)^2 = (\Delta P)^2 = \frac{5}{8} = \frac{F_5}{F_6}.$$

This suggests to construct the sequence of states with ratio of Fibonacci numbers for any n. The sequence of circles

$$|\zeta_n|^2 = \frac{F_{n-1}}{F_{n-2}} - \frac{1}{2}$$

in complex plane ζ, determine dispersions

$$(\Delta X_n)^2 = (\Delta P_n)^2 = \frac{F_n}{F_{n+1}},$$

Uncertainty Relations and Fibonacci Sequence

and uncertainty relations

$$\Delta X_n \Delta P_n = \frac{F_n}{F_{n+1}}.$$

The radius square of circles $|\zeta_n|^2$ is Fibonacci oscillating around the value, corresponding to the limit $n \to \infty$,

$$|\zeta_\infty|^2 = \varphi - \frac{1}{2},$$

where φ is the Golden Ratio.

Uncertainty Relations and Fibonacci Sequence

In this limit we have the Golden dispersions

$$(\Delta X_\infty)^2 = (\Delta P_\infty)^2 = \frac{1}{\varphi},$$

and the Golden uncertainty relation

$$\Delta X_\infty \Delta P_\infty = \frac{1}{\varphi}.$$

Super-Quantum Computer ?

By using higher super-particle states it is possible to construct super-qudit states as an extension of the super-qubit.
Super-qubit state as unit of quantum information is determined by product of two unit (Bloch) spheres

$$S^2 \times S^2$$

much bigger capacity than usual qubit state $\rightarrow$
super-qubit gates,
entanglement of N super-qubit states,
algorithms,
supergravity, etc

Super-Qubit and Schrödinger super-cat states

Displaced super-qubit state

$$\mathcal{D}(\alpha)|\Psi\rangle = \cos\frac{\theta}{2}|0,\zeta;\alpha\rangle + \sin\frac{\theta}{2}e^{i\phi}|1,\zeta;\alpha\rangle$$

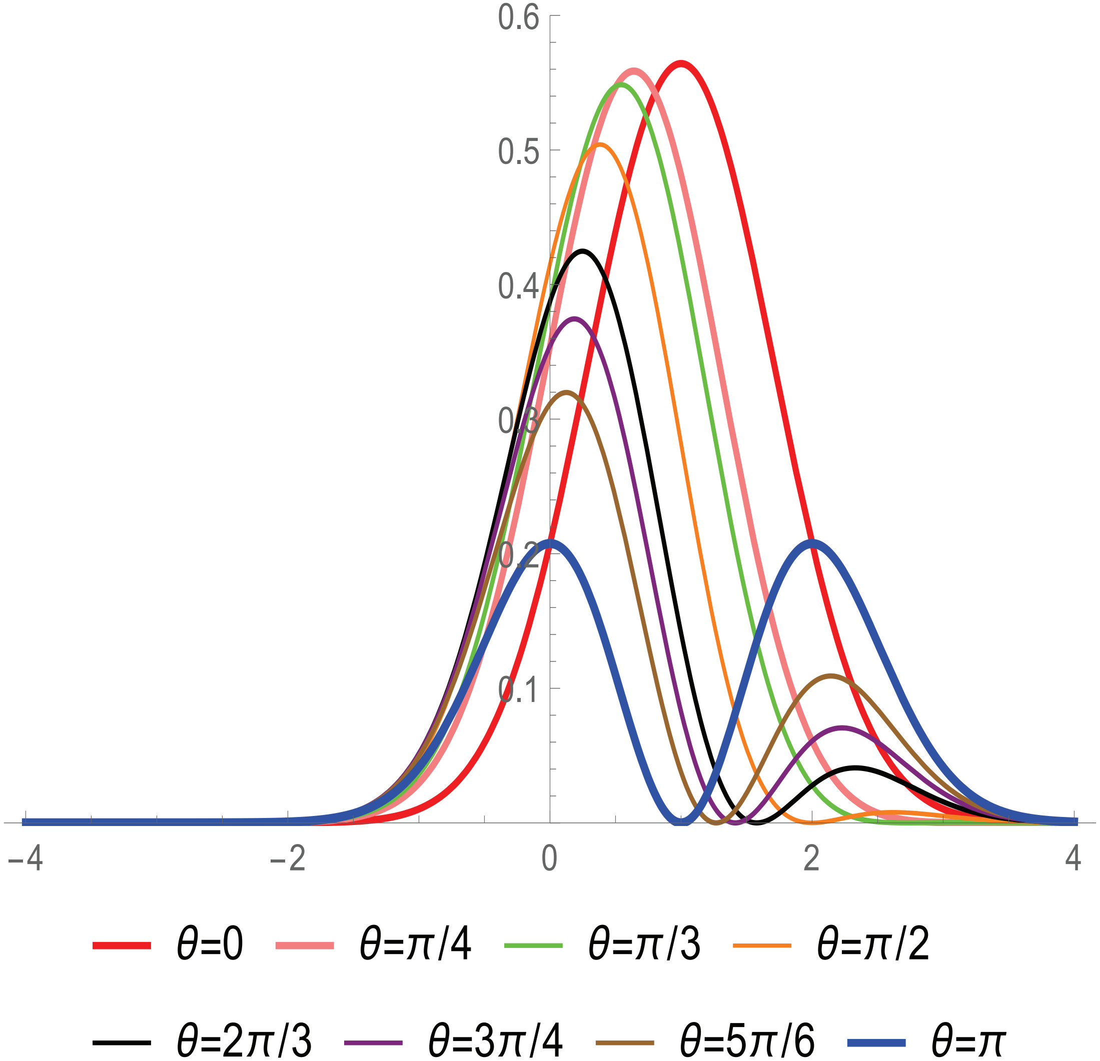
the Schrödinger Super-cat states - macroscopic, orthogonal and entangled

$\rightarrow$ Super-coherent states

Wave function of Super-cat states

$C = 0 \rightarrow \theta = 0$ separable Gaussian cat state (classical state)

$C = 1 \rightarrow \theta = \pi$ maximally entangled non-classical cat state



7. The pq-deformed Quantum Oscillator

The pq-deformed Quantum Oscillator

Here, we briefly recall basic properties of two parametric, the pq-deformed quantum oscillator. It is determined by the pair of creation and annihilation operators, $a^\dagger_{p,q}$, $a_{p,q}$, and the number operator N, satisfying the quantum algebra

$$a_{p,q}a^\dagger_{p,q} - pa^\dagger_{p,q}a_{p,q} = q^N,$$
$$a_{p,q}a^\dagger_{p,q} - qa^\dagger_{p,q}a_{p,q} = p^N,$$

$$[N, a^\dagger_{pq}] = a^\dagger_{pq}$$
$$[N, a_{pq}] = -a_{pq}$$

The pq deformed number operator

The pq-number operator is determined by formula

$$[N]_{p,q} = \frac{p^N - q^N}{p - q}$$

and it can be represented in factorized form as

$$[N]_{p,q} = a^\dagger_{p,q} a_{p,q} \qquad [N+I]_{p,q} = a_{p,q} a^\dagger_{p,q}$$

The operator satisfies recursion relations

$$\begin{aligned} [N+I]_{p,q} &= (p+q)[N]_{p,q} - pq[N-I]_{p,q}, \\ [N+I]_{p,q} &= p[N]_{p,q} + q^N = q[N]_{p,q} + p^N, \\ p^N &= p[N]_{p,q} - pq[N-I]_{p,q}, \\ q^N &= q[N]_{p,q} - pq[N-I]_{p,q} \end{aligned}$$

The pq deformed number operator

gives commutator

$$a_{p,q}a^{\dagger}_{p,q} - a^{\dagger}_{p,q}a_{p,q} = [a_{p,q}, a^{\dagger}_{p,q}] = [N+I]_{p,q} - [N]_{p,q}$$

The Fock space basis states $\{|n\rangle_{p,q}\}$, defined as

$$|n\rangle_{p,q} = \frac{(a^{\dagger}_{p,q})^n}{\sqrt{[n]_{p,q}!}}|0\rangle_{p,q}, \qquad a_{p,q}|0\rangle_{p,q} = 0,$$

are the orthonormal ${}_{p,q}\langle n|m\rangle_{p,q} = \delta_{nm}$ eigenstates of $[N]_{p,q}$,

$$[N]_{p,q}|n\rangle_{p,q} = [n]_{pq}|n\rangle_{p,q}\ ,$$

and satisfy relations

$$a^{\dagger}_{p,q}|n\rangle_{p,q} = \sqrt{[n+1]_{p,q}}\,|n{+}1\rangle_{p,q}, \qquad a_{p,q}|n\rangle_{p,q} = \sqrt{[n]_{p,q}}\,|n{-}1\rangle_{p,q}\,.$$

Hamiltonian and spectrum

operators $a^\dagger_{p,q}$, $a_{p,q}$ are connected with bosonic operators $a^\dagger$, a by nonlinear transformation

$$a_{p,q} = a\sqrt{\frac{[N]_{p,q}}{N}} = \sqrt{\frac{[N+I]_{p,q}}{N+I}}\,a$$

The spectrum of pq-deformed bosonic Hamiltonian

$$H_{p,q} = \frac{\hbar\omega}{2}(a_{p,q}a^\dagger_{p,q} + a^\dagger_{p,q}a_{p,q}) = \frac{\hbar\omega}{2}([N]_{p,q} + [N+I]_{p,q})$$

is not equidistant and it is determined by sequence of pq-numbers

$$E_n = \frac{\hbar\omega}{2}([n]_{p,q} + [n+1]_{p,q}), \quad n = 0, 1, 2, ...$$

Quantum group structure and root of unity

Special case $p \to q^2$, $q \to q^{-2}$

primitive root of unity $q^{2n} = 1 \to q = e^{i\frac{\pi}{n}}$

Clock and shift matrices

The dilatation (rotation) operator in Fock space

$$q^{2N} = I \otimes \Sigma_3 = I \otimes \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & q^2 & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & q^{2(n-1)} \end{pmatrix}$$

is related with Sylvester clock and shift matrices

$$\Sigma_3 = H \Sigma_1^\dagger H^\dagger$$

Sylvester Clock and Shift matrices

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & 0 \\ . & . & ... & . & \\ 0 & 0 & ... & 1 & 0 \end{pmatrix} H = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & \bar{q}^2 & ... & \bar{q}^{2(n-1)} \\ . & . & ... & . \\ 1 & \bar{q}^{2(n-1)} & ... & \bar{q}^{2(n-1)^2} \end{pmatrix}$$

The matrices are q^2-commutative

$$\Sigma_1 \Sigma_3 = q^2 \, \Sigma_3 \Sigma_1$$

and satisfy

$$\Sigma_1^n = I, \qquad \Sigma_3^n = I$$

Symmetric quantum calculus and root of unity

From dilatation operator q^{2N} we have q^2-number operator

$$[N]_{\tilde{q}^2} = \frac{q^{2N} - q^{-2N}}{q^2 - q^{-2}} = I \otimes diag([0]_{\tilde{q}^2}, [1]_{\tilde{q}^2}, ..., [n-1]_{\tilde{q}^2})$$

for the symmetric calculus, as matrix with diagonal elements given by q-numbers: $[n]_{\tilde{q}^2} = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}$.
The operator can be factorized as

$$[N]_{\tilde{q}^2} = B^+ B, \qquad [N+1]_{\tilde{q}^2} = BB^+$$

where

$$B^n = 0, \qquad (B^+)^n = 0$$

and

$$B = I \otimes a \sqrt{\frac{[N]_{\tilde{q}^2}}{N}}.$$

Explicitly in matrix form it is

$$B = I \otimes \begin{pmatrix} 0 & \sqrt{[1]} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{[2]} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$B^{+} = I \otimes \begin{pmatrix} 0 & 0 & \dots & 0 \\ \sqrt{[1]} & 0 & \dots & 0 \\ 0 & \sqrt{[2]} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Finite Discrete Spectrum

$p \to q^2$, $q \to q^{-2}$ The operators satisfy quantum algebra

$$
\begin{aligned}
BB^+ - q^2 B^+ B &= q^{-2N} \\
BB^+ - q^{-2} B^+ B &= q^{2N}
\end{aligned}
$$

and determine quantum q^2-oscillator with Hamiltonian

$$H = \frac{\hbar\omega}{2}\left([N]_{\tilde{q}^2} + [N+I]_{\tilde{q}^2}\right)$$

Eigenstates of number operator

$$q^{2N}|\psi_k\rangle = q^{2k}|\psi_k\rangle$$

$$[N]_{\tilde{q}^2}|\psi_k\rangle = [k]_{\tilde{q}^2}|\psi_k\rangle,$$

Special Case: Symmetric q-calculus - root of unity

Eigenstates of Hamiltonian

$$H|\psi_k\rangle = E_k|\psi_k\rangle = \frac{\hbar\omega}{2}\left([k]_{\tilde{q}^2} + [k+1]_{\tilde{q}^2}\right)|\psi_k\rangle,$$

with finite spectrum, $k = 0, 1, ..., n-1$,

$$E_k = \frac{\hbar\omega}{2}\frac{\sin\frac{2\pi}{n}\left(k+\frac{1}{2}\right)}{\sin\frac{\pi}{n}}.$$

1. H. Weyl "The Theory of Groups and Quantum Mechanics" (1931) proposed clock and shift matrices for quantum mechanics of finite dimensional systems
2. The spectrum was obtained also for description of two anyons (Floratos, Tomaras 1990).

The pq-Coherent States

The pq-coherent states are defined as eigenstates of annihilation operator $a_{p,q}$,

$$a_{p,q}|\alpha\rangle_{p,q} = \alpha|\alpha\rangle_{p,q}, \quad \alpha \in C. \tag{23}$$

The coherent states (not normalized), expanded in the deformed basis states take the form

$$|\alpha\rangle_{p,q} = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_{p,q}!}}|n\rangle_{p,q} = e_{p,q}^{\alpha a_{p,q}^{\dagger}}|0\rangle_{p,q}, \tag{24}$$

and have the inner products

$${}_{p,q}\langle\beta|\alpha\rangle_{p,q} = e_{p,q}^{\bar{\beta}\alpha}, \qquad {}_{p,q}\langle\alpha|\alpha\rangle_{p,q} = e_{p,q}^{|\alpha|^2}. \tag{25}$$

The pq-Coherent States

The normalized states are

$$|0,\alpha\rangle_{p,q} = \left(e_{p,q}^{|\alpha|^2}\right)^{-1/2} |\alpha\rangle_{p,q} = \left(e_{p,q}^{|\alpha|^2}\right)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_{p,q}!}} |n\rangle_{p,q},$$

with the following inner product

$$_{p,q}\langle 0,\alpha|0,\beta\rangle_{p,q} = \frac{{e_{p,q}}^{\bar{\alpha}\beta}}{\sqrt{{e_{p,q}}^{|\alpha|^2}\, {e_{p,q}}^{|\beta|^2}}}\,.$$

Coordinate-momentum uncertainty relations

In terms of pq-deformed coordinate and momentum operators, defined as

$$X_{p,q} = \sqrt{\frac{\hbar}{2m\omega}}(a^{\dagger}_{p,q} + a_{p,q}), \qquad P_{p,q} = i\sqrt{\frac{m\hbar\omega}{2}}(a^{\dagger}_{p,q} - a_{p,q})$$

the Hamiltonian is

$$H_{p,q} = \frac{1}{2m}P^2_{p,q} + \frac{m\omega^2}{2}X^2_{p,q}$$

Proposition

$$\frac{{}_{p,q}\langle\alpha|X_{p,q}|\alpha\rangle_{p,q}}{{}_{p,q}\langle\alpha|\alpha\rangle_{p,q}} = \sqrt{\frac{2\hbar}{m\omega}}\,\Re\alpha,\ \frac{{}_{p,q}\langle\alpha|P_{p,q}|\alpha\rangle_{p,q}}{{}_{p,q}\langle\alpha|\alpha\rangle_{p,q}} = \sqrt{2m\hbar\omega}\,\Im\alpha,$$

$$\frac{{}_{p,q}\langle\alpha|X^2_{p,q}|\alpha\rangle_{p,q}}{{}_{p,q}\langle\alpha|\alpha\rangle_{p,q}} = \frac{\hbar}{2m\omega}\left(\alpha^2+\bar{\alpha}^2+(1+q)|\alpha|^2+\frac{e_{p,q}^{p|\alpha|^2}}{e_{p,q}^{|\alpha|^2}}\right),$$

$$\left(\frac{{}_{p,q}\langle\alpha|X_{p,q}|\alpha\rangle_{p,q}}{{}_{p,q}\langle\alpha|\alpha\rangle_{p,q}}\right)^2 = \frac{\hbar}{2m\omega}\left(\alpha^2+\bar{\alpha}^2+2|\alpha|^2\right),$$

$$\frac{{}_{p,q}\langle\alpha|P^2_{p,q}|\alpha\rangle_{p,q}}{{}_{p,q}\langle\alpha|\alpha\rangle_{p,q}} = \frac{m\hbar\omega}{2}\left(-\alpha^2-\bar{\alpha}^2+(1+q)|\alpha|^2+\frac{e_{p,q}^{p|\alpha|^2}}{e_{p,q}^{|\alpha|^2}}\right),$$

$$\left(\frac{{}_{p,q}\langle\alpha|P_{p,q}|\alpha\rangle_{p,q}}{{}_{p,q}\langle\alpha|\alpha\rangle_{p,q}}\right)^2 = \frac{m\hbar\omega}{2}\left(-\alpha^2-\bar{\alpha}^2+2|\alpha|^2\right).$$

Dispersion

Dispersions for deformed coordinate and momentum are defined as

$$\langle \Delta X^2 \rangle = \frac{{}_{p,q}\langle \alpha | X_{p,q}^2 | \alpha \rangle_{p,q}}{{}_{p,q}\langle \alpha | \alpha \rangle_{p,q}} - \left(\frac{{}_{p,q}\langle \alpha | X_{p,q} | \alpha \rangle_{p,q}}{{}_{p,q}\langle \alpha | \alpha \rangle_{p,q}} \right)^2,$$

$$\langle \Delta P^2 \rangle = \frac{{}_{p,q}\langle \alpha | P_{p,q}^2 | \alpha \rangle_{p,q}}{{}_{p,q}\langle \alpha | \alpha \rangle_{p,q}} - \left(\frac{{}_{p,q}\langle \alpha | P_{p,q} | \alpha \rangle_{p,q}}{{}_{p,q}\langle \alpha | \alpha \rangle_{p,q}} \right)^2.$$

Theorem

$$\langle \Delta X^2 \rangle = \frac{\hbar}{2m\omega}\left((q-1)|\alpha|^2 + \frac{e_{p,q}^{p|\alpha|^2}}{e_{p,q}^{|\alpha|^2}}\right)$$
$$\langle \Delta P^2 \rangle = \frac{m\hbar\omega}{2}\left((q-1)|\alpha|^2 + \frac{e_{p,q}^{p|\alpha|^2}}{e_{p,q}^{|\alpha|^2}}\right)$$

and

$$\Delta X \Delta P = \frac{\hbar}{2}\left((q-1)|\alpha|^2 + \frac{e_{p,q}^{p|\alpha|^2}}{e_{p,q}^{|\alpha|^2}}\right)$$

In these formulas equivalent expressions appear by interchanging of p and q.

Theorem

Example 1: For non-symmetric, $p = 1$ case,

$$\Delta X \Delta P = \frac{\hbar}{2}\left(1 + (q-1)|\alpha|^2\right)$$

Theorem

For small values of $|\alpha|^2 << 1$, we have

$$\Delta X \Delta P \approx \frac{\hbar}{2}\left(1 + (p + q - 2)|\alpha|^2\right)$$

In Fibonacci case $p = \varphi$, $q = \varphi'$, this gives uncertainty

$$\Delta X \Delta P \approx \frac{\hbar}{2}\left(1 - |\alpha|^2\right)$$

smaller than the classical one $\hbar/2$.
In the case of Fibonacci divisors $p = \varphi^k$, $q = {\varphi'}^k$, the deviation is determined by Lucas numbers L_k,

$$\Delta X \Delta P \approx \frac{\hbar}{2}\left(1 + (L_k - 2)|\alpha|^2\right)$$

8. The pq-Deformed Super-symmetric Oscillator

The pq-Deformed Super-symmetric Oscillator

Two operators

$$Q_{p,q} = \begin{pmatrix} 0 & 0 \\ a_{p,q} & 0 \end{pmatrix}, \quad Q^{\dagger}_{p,q} = \begin{pmatrix} 0 & a^{\dagger}_{p,q} \\ 0 & 0 \end{pmatrix}$$

determine Hamiltonian of the pq-deformed supersymmetric oscillator,

$$H^S_{p,q} = \frac{\hbar\omega}{2}(Q_{p,q}, Q^{\dagger}_{p,q} + Q^{\dagger}_{p,q}Q_{p,q}) = \frac{\hbar\omega}{2} \begin{pmatrix} [N]_{p,q} & 0 \\ 0 & [N+I]_{p,q} \end{pmatrix}$$

The pq-Deformed Super-symmetric Oscillator

The pq-deformed super-number operator, defined as

$$[\mathcal{N}]_{pq} = \begin{pmatrix} [N]_{p,q} & 0 \\ 0 & [N+I]_{p,q} \end{pmatrix}$$

is connected with the supersymmetric number operator $\mathcal{N} = I_f \otimes N + N_f \otimes I_{p,q}$ by formula

$$[\mathcal{N}]_{p,q} = \frac{p^{\mathcal{N}} - q^{\mathcal{N}}}{p - q}$$

The Hamiltonian is expressed in terms of it as

$$H^S_{p,q} = \frac{\hbar\omega}{2}[\mathcal{N}]_{p,q}$$

Spectrum

after taking partial trace in fermionic variables it gives the Hamiltonian for the pq-deformed oscillator

$$Tr_f H_{p,q}^S = \frac{\hbar\omega}{2}([N]_{p,q} + [N+I]_{p,q}) = H_{p,q}$$

The eigenstates of operators, $[\mathcal{N}]_{p,q}$ and $H_{p,q}^S$ are the same, with eigenvalues given by the pq-number sequence $[n]_{p,q}$, so that the spectrum of energy is

$$E_n = \frac{\hbar\omega}{2}[n]_{p,q}$$

The pq-Deformed Super-Number States

The eigenstates of operator $[\mathcal{N}]_{p,q}$ are of two types,

$$[\mathcal{N}]_{p,q} \begin{pmatrix} |n\rangle_{p.q} \\ 0 \end{pmatrix} = [n]_{p,q} \begin{pmatrix} |n\rangle_{p,q} \\ 0 \end{pmatrix}$$

and

$$[\mathcal{N}]_{p,q} \begin{pmatrix} 0 \\ |n-1\rangle_{p,q} \end{pmatrix} = [n]_{p,q} \begin{pmatrix} 0 \\ |n-1\rangle_{p,q} \end{pmatrix}$$

with number of fermions equal to zero and to one, correspondingly. These states are separable, while an arbitrary superposition of the states is also an eigenstate, but it could be entangled.

The pq-Deformed Super-Number States

The normalized pq-deformed super-number state (up to the global phase), is the double degenerate superposition of the above states,

$$|n;\theta,\phi\rangle_{p,q} = \cos\frac{\theta}{2}\begin{pmatrix} |n\rangle_{p,q} \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix} 0 \\ |n-1\rangle_{p,q} \end{pmatrix}$$

It is an eigenstate of $[\mathcal{N}]_{p,q}$, giving the energy level

$$E_n = \frac{\hbar\omega}{2}[n_f + n_b]_{p,q}$$

where $n = n_f + n_b$ counts number of superparticles in the state.

The pq-Deformed Super-Coherent States

Super-annihilation operators are defined as

$$A = \begin{pmatrix} p\, a_{p,q} & -1 \\ 0 & q\, a_{p,q} \end{pmatrix}, \qquad A^T = \begin{pmatrix} p\, a_{p,q} & 0 \\ -1 & q\, a_{p,q} \end{pmatrix}$$

Eigenstates of these operators we call as the pq-deformed super-coherent states

The pq-Deformed Super-Coherent States

Proposition

The separable pq-deformed super-coherent states are

$$|\alpha, sep \uparrow\rangle = (e_{p,q}^{\frac{|\alpha|^2}{p^2}})^{-1/2} \begin{pmatrix} |\frac{\alpha}{p}\rangle \\ 0 \end{pmatrix}$$

$$|\alpha, sep \downarrow\rangle = (e_{p.q}^{\frac{|\alpha|^2}{q^2}})^{-1/2} \begin{pmatrix} 0 \\ |\frac{\alpha}{q}\rangle \end{pmatrix}$$

$$A|\alpha, sep \uparrow\rangle = \alpha|\alpha, sep \uparrow\rangle$$

$$A^T|\alpha, sep \downarrow\rangle = \alpha|\alpha, sep \downarrow\rangle$$

The coordinate and momentum uncertainty relations

The pq-deformed supersymmetric coordinate and momentum operators are defined as, $(\hbar = m = \omega = 1)$,

$$X_{p,q} = I_f \otimes \frac{a^\dagger_{p,q} + a_{p,q}}{\sqrt{2}}, \qquad P_{p,q} = I_f \otimes i\frac{a^\dagger_{p,q} - a_{p,q}}{\sqrt{2}}$$

Proposition

$$(\Delta X_{p,q})^2 = (\Delta P_{p,q})^2 = \Delta X_{p,q}\Delta P_{p,q} = \frac{1}{2}\left((q-1)\frac{|\alpha|^2}{p^2} + \frac{e_{p,q}^{p\frac{|\alpha|^2}{p^2}}}{e_{p,q}^{\frac{|\alpha|^2}{p^2}}}\right)$$

$$(\Delta X_{p,q})^2 = (\Delta P_{p,q})^2 = \Delta X_{p,q}\Delta P_{p,q} = \frac{1}{2}\left((p-1)\frac{|\alpha|^2}{q^2} + \frac{e_{p,q}^{q\frac{|\alpha|^2}{q^2}}}{e_{p,q}^{\frac{|\alpha|^2}{q^2}}}\right)$$

The coordinate and momentum uncertainty relations

Comparison of these formulas for separable supersymmetric states with the ones for pure pq-deformed states shows that they coincide after transformation

$$\alpha \to \alpha/p$$

and

$$\alpha \to \alpha/q$$

respectively

Entangled normalized deformed super-coherent states

$$|\alpha, L\rangle = \frac{1}{\sqrt{\frac{|\alpha|^2}{p^4 q} e_{p,q}^{\frac{|\alpha|^2}{p^2 q^2}} + \frac{1}{p^2} e_{p,q}^{\frac{|\alpha|^2}{pq^2}} + e_{p,q}^{\frac{|\alpha|^2}{q^2}}}} \begin{pmatrix} q|\frac{\alpha'}{pq}\rangle_{p,q} \\ |\frac{\alpha}{q}\rangle_{p,q} \end{pmatrix}$$

satisfies equation

$$A|\alpha, L\rangle = \alpha|\alpha, L\rangle$$

$$|\alpha, B\rangle = \frac{1}{\sqrt{\frac{|\alpha|^2}{p^2 q^3} e_{p,q}^{\frac{|\alpha|^2}{p^2 q^2}} + \frac{1}{q^2} e_{p,q}^{\frac{|\alpha|^2}{pq^2}} + e_{p,q}^{\frac{|\alpha|^2}{p^2}}}} \begin{pmatrix} |\frac{\alpha}{p}\rangle \\ p|\frac{\alpha'}{pq}\rangle \end{pmatrix}$$

is subject to the eigenvalue problem

$$A^T|\alpha, B\rangle = \alpha|\alpha, B\rangle$$

Proof

Representing eigenstate of A in the form,
$|A\rangle = |0\rangle_f |\psi_0\rangle_{p,q} + |1\rangle_f |\psi_1\rangle_{p,q}$, we get equations

$$a_{p,q}|\psi_1\rangle_{p,q} = \frac{\alpha}{q}|\psi_1\rangle_{p,q}, \qquad (26)$$

$$(a_{p,q} - \frac{\alpha}{p})|\psi_0\rangle_{p,q} = \frac{1}{p}|\psi_1\rangle_{p,q}. \qquad (27)$$

Solution of the first one is just the pq-deformed coherent state, $|\psi_1\rangle_{p,q} = \lambda_1|\frac{\alpha}{q}\rangle_{p,q}$. For the second one, the general solution $|\psi_0\rangle_{p,q} = |\psi_0\rangle_{hom} + |\psi_0\rangle_{nonhom}$ is superposition of general solution for homogeneous part and particular solution for the non-homogeneous one. For the former, $|\psi_0\rangle_{hom} = \lambda_0|\frac{\alpha}{p}\rangle_{p,q}$ and for the second one, $|\psi_0\rangle_{nonhom} = \lambda_1 q|\frac{\alpha'}{pq}\rangle_{p,q}$.

Proof

Thus,

$$|A\rangle = \lambda_0 |0\rangle_f \otimes \left| \frac{\alpha}{p} \right\rangle_{p,q} + \lambda_1 \left(|0\rangle_f \otimes q \left| \frac{\alpha'}{pq} \right\rangle_{p,q} + |1\rangle_f \otimes \left| \frac{\alpha}{q} \right\rangle_{p,q} \right)$$

Here, arbitrary coefficients λ_0 and λ_1 can be chosen to normalize the state and to produce two orthogonal states.

Entanglement of pq Super Coherent States

Proposition

For a generic normalized state

$$|\Psi\rangle = \sum_{k=0}^{\infty} c_{0k}|0\rangle_f \otimes |k\rangle_{p,q} + \sum_{k=0}^{\infty} c_{1k}|1\rangle_f \otimes |k\rangle_{p,q} =$$

$$|0\rangle_f \otimes |\psi_0\rangle_{p,q} + |1\rangle_f \otimes |\psi_1\rangle_{p,q}$$

from $\mathcal{H}_f \otimes \mathcal{H}_{p,q}$ Hilbert space, the concurrence is

$$C = 2\sqrt{det \begin{pmatrix} {}_{p,q}\langle\psi_0|\psi_0\rangle_{p,q} & {}_{p,q}\langle\psi_0|\psi_1\rangle_{p,q} \\ \\ {}_{p,q}\langle\psi_1|\psi_0\rangle_{p,q} & {}_{p,q}\langle\psi_1|\psi_1\rangle_{p,q} \end{pmatrix}}$$

Entanglement of pq Super Coherent States

Proposition

For the pq-deformed super-number state

$$|n;\theta,\phi\rangle_{p,q} = \cos\frac{\theta}{2}\begin{pmatrix} |n\rangle_{p,q} \\ 0 \end{pmatrix} + \sin\frac{\theta}{2}e^{i\phi}\begin{pmatrix} 0 \\ |n-1\rangle_{p,q} \end{pmatrix}$$

the concurrence is

$$C = \sin\theta$$

Theorem: concurrence for pq-deformed supersymmetric states

$$C_L(|\alpha|^2) = 2N_L^2\sqrt{\frac{1}{p^2}e_{p,q}^{\frac{|\alpha|^2}{q^2}}\, e_{p,q}^{\frac{|\alpha|^2}{pq^2}} + \frac{|\alpha|^2}{p^4 q}e_{p,q}^{\frac{|\alpha|^2}{q^2}}\, e_{p,q}^{\frac{|\alpha|^2}{p^2q^2}} - \frac{|\alpha|^2}{p^2q^2}\left(e_{p,q}^{\frac{|\alpha|^2}{pq^2}}\right)^2},$$

$$N_L^{-2} = \frac{|\alpha|^2}{p^4 q}e_{p,q}^{\frac{|\alpha|^2}{p^2q^2}} + \frac{1}{p^2}e_{p,q}^{\frac{|\alpha|^2}{pq^2}} + e_{p,q}^{\frac{|\alpha|^2}{q^2}}.$$

$$C_B(|\alpha|^2) = 2N_B^2\sqrt{\frac{1}{q^2}e_{p,q}^{\frac{|\alpha|^2}{p^2}}\, e_{p,q}^{\frac{|\alpha|^2}{pq^2}} + \frac{|\alpha|^2}{p^2 q^3}e_{p,q}^{\frac{|\alpha|^2}{p^2}}\, e_{p,q}^{\frac{|\alpha|^2}{p^2q^2}} - \frac{|\alpha|^2}{p^2q^2}\left(e_{p,q}^{\frac{|\alpha|^2}{p^2q}}\right)^2},$$

$$N_B^{-2} = \frac{|\alpha|^2}{p^2 q^3}e_{p,q}^{\frac{|\alpha|^2}{p^2q^2}} + \frac{1}{q^2}e_{p,q}^{\frac{|\alpha|^2}{pq^2}} + e_{p,q}^{\frac{|\alpha|^2}{p^2}}.$$

Proposition

In the limit $\alpha \to 0$, entangled states become the reference states

$$|0, L\rangle = \frac{1}{\sqrt{1+p^2}} \begin{pmatrix} |1\rangle_{p,q} \\ p|0\rangle_{p,q} \end{pmatrix}$$

with concurrence

$$C_L = \frac{2|p|}{1+p^2}$$

and

$$|0, B\rangle = \frac{1}{\sqrt{1+q^2}} \begin{pmatrix} q|0\rangle_{p,q} \\ |1\rangle_{p,q} \end{pmatrix}$$

with concurrence

$$C_B = \frac{2|q|}{1+q^2},$$

Theorem

The coordinate-momentum uncertainty relations in entangled reference states are

$$(\Delta X_{p,q})^2_{0L} = (\Delta P_{p,q})^2_{0L} = (\Delta X_{p,q}\Delta P_{p,q})_{0L} = \frac{1}{2}\left(1+\frac{p+q}{1+p^2}\right)$$

$$(\Delta X_{p,q})^2_{0B} = (\Delta P_{p,q})^2_{0B} = (\Delta X_{p,q}\Delta P_{p,q})_{0B} = \frac{1}{2}\left(1+\frac{p+q}{1+q^2}\right)$$

Conclusions

We found that for pq-deformed quantum oscillator, the uncertainty relations depend on deformation parameters p and q. In supersymmetric case, we constructed entangled coherent states with concurrence depending on these parameters as well. This imply that deformation parameters introduce non-classicality to the coherent states in both cases.