Modern Techniques to Compute Scattering Amplitudes and Feynman Integrals

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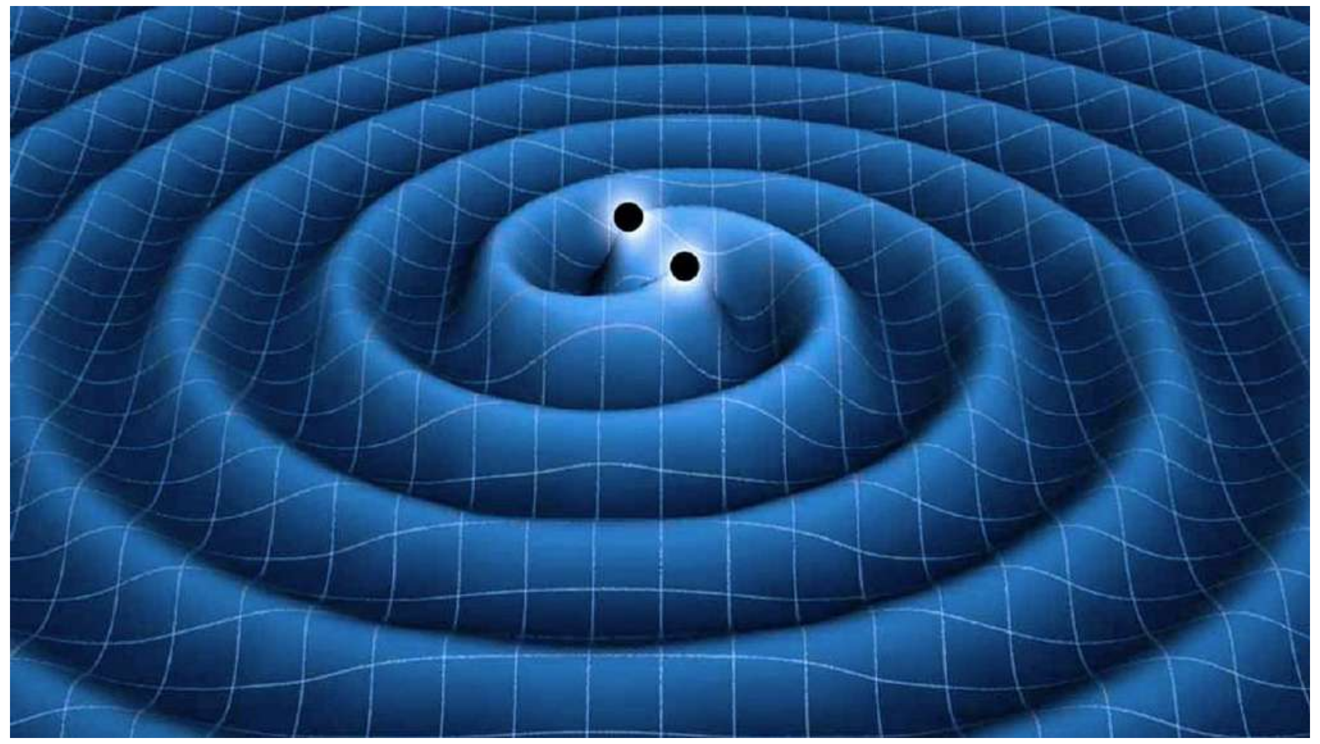
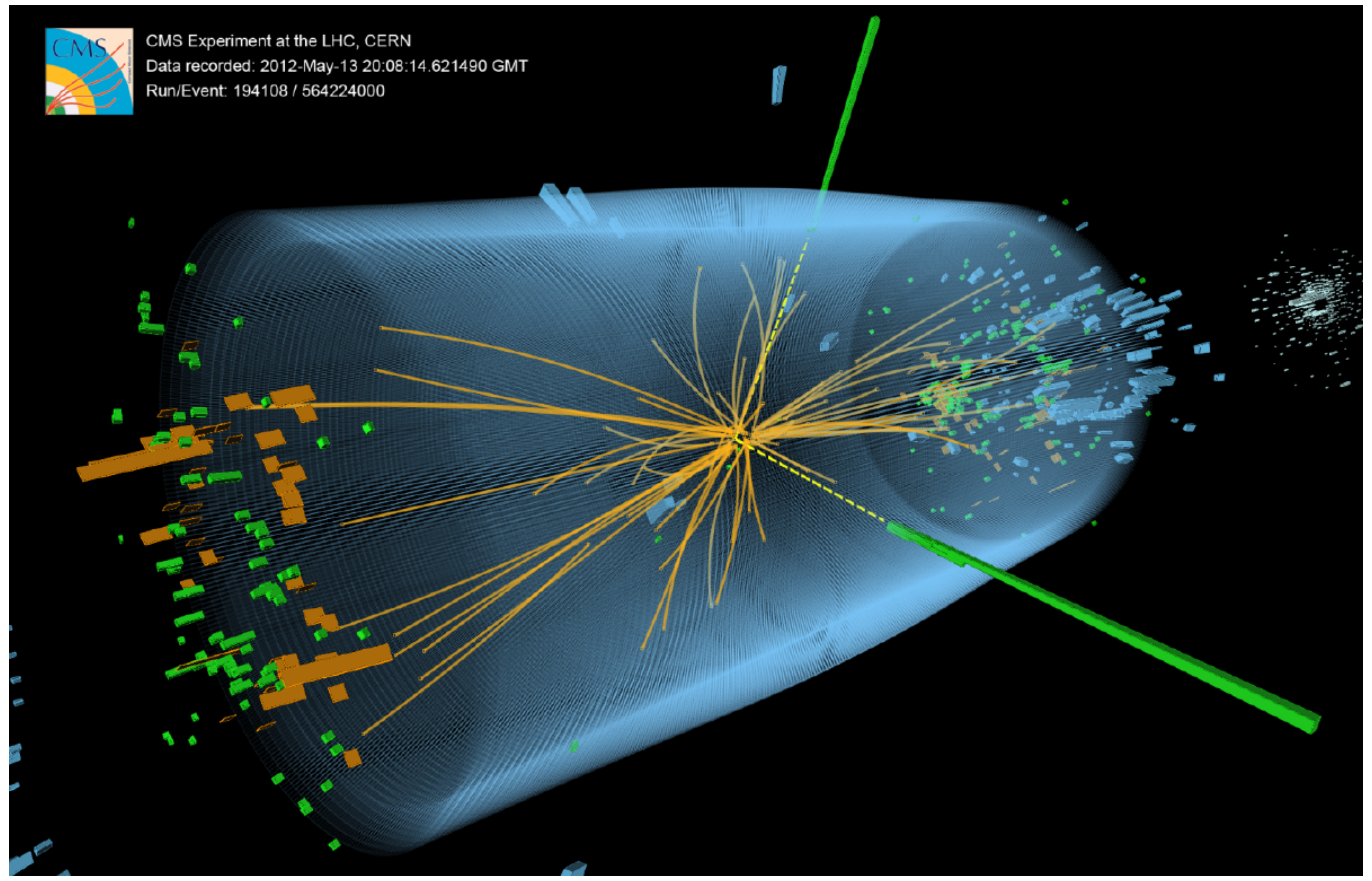
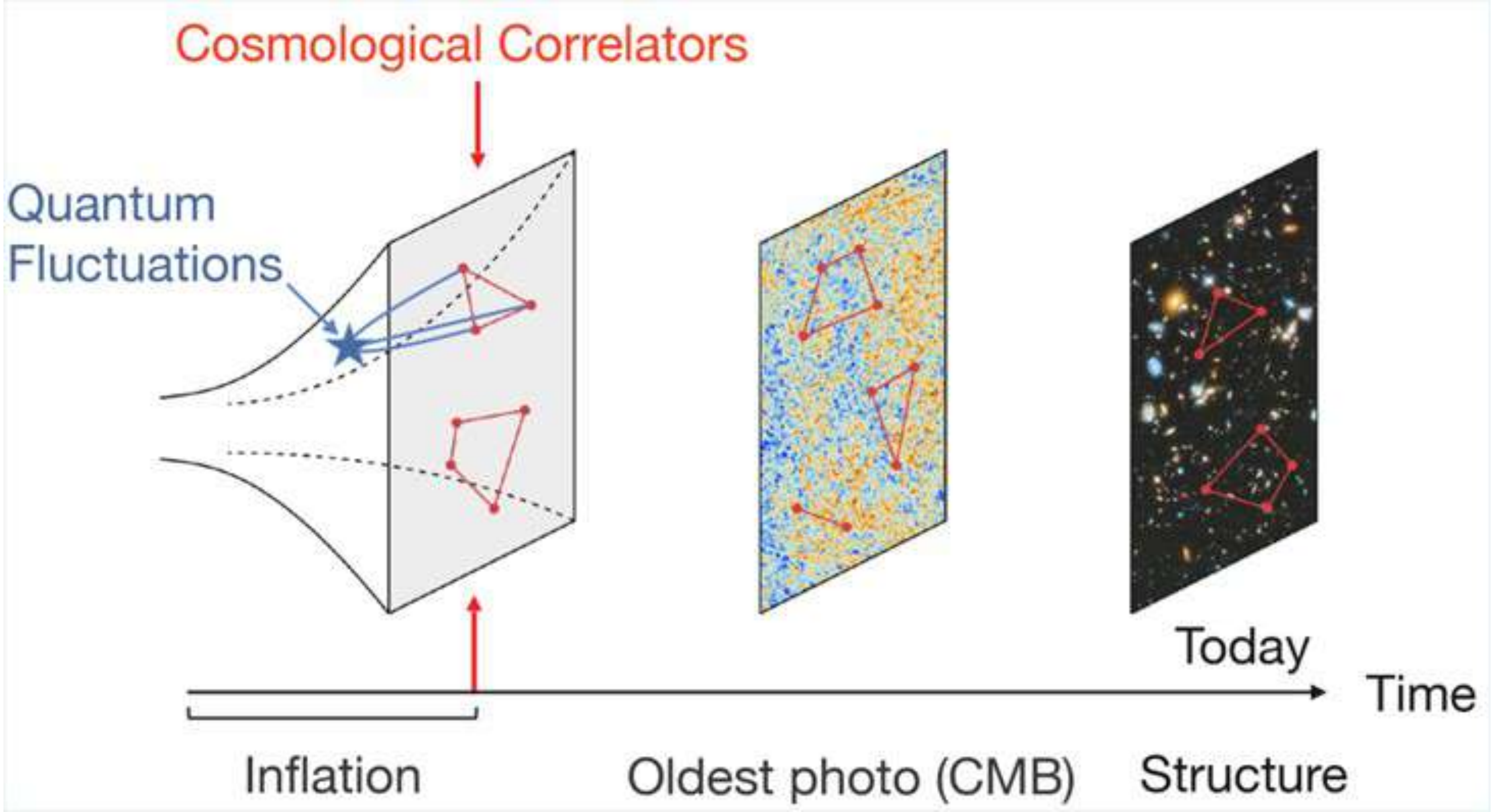
Motivation

1. Phenomenological motivation







Motivation

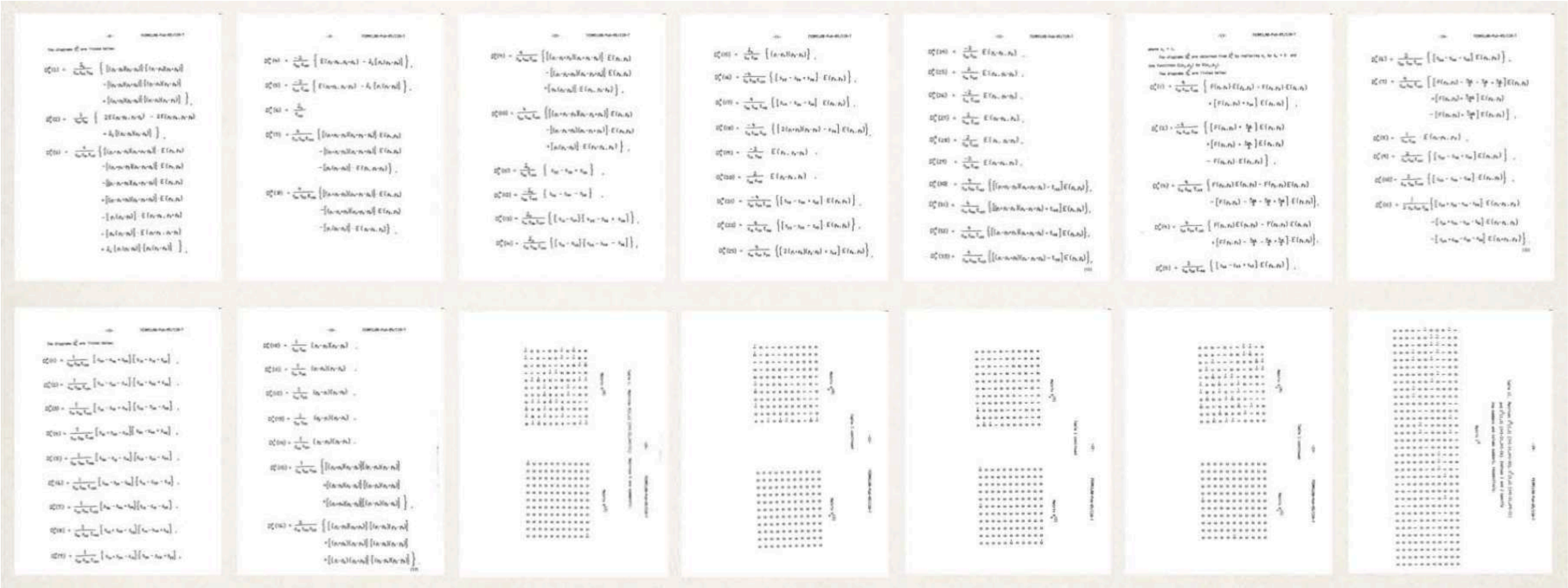
2. Analytic motivation

Scattering amplitudes is much more simpler than Feynman integrals!

Textbook procedure : **Draw all the Feynman diagrams** $\longrightarrow$ **Evaluate the diagrams with Feynman rules** $\longrightarrow$ **Sum all**

Process : $gg \to gggg$

220 Feynman diagram, ~100 pages of calculations



[Parke, Taylor, 1985]

combining all $\longrightarrow$

$$\mathcal{M}_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle^3}$$

[Parke, Taylor, 1986]

Huge cancellation!

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$

$$\langle 12 \rangle = \epsilon_{ab} \lambda^{(1)}_a \lambda^{(2)}_b$$

$$[12] = \epsilon_{ab} \tilde{\lambda}^{(1)}_{\dot{a}} \tilde{\lambda}^{(2)}_{\dot{b}}$$

“Birth of amplitudes”

Motivation

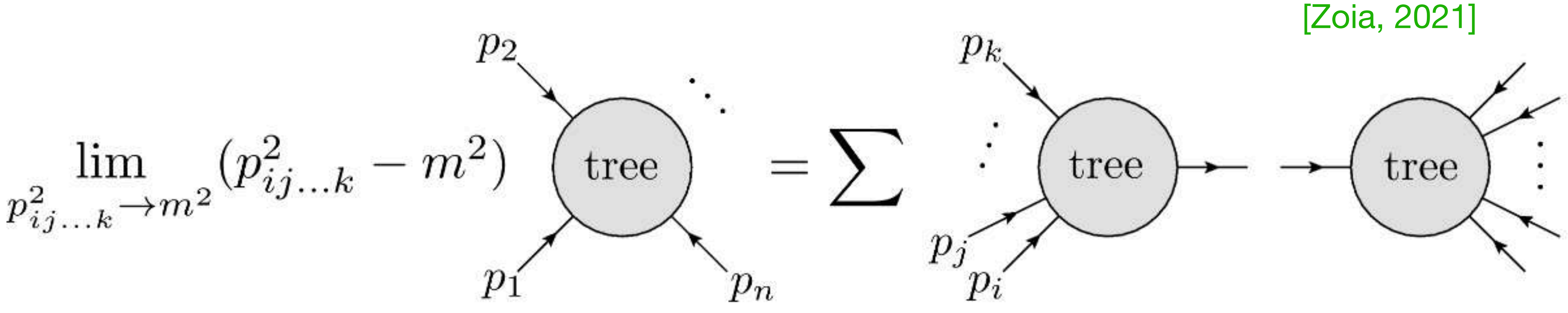
2. Analytic motivation

Tree-level

BCFW recursion relation is an efficient way to compute higher-point tree-level amplitudes from lower-point ones using the structure of the poles. [Britto, Cachazo, Feng, Witten, 2005]

$$A_n(1,\ldots,n) = \sum_{i=3}^{n-1} \sum_{h=\pm} A_i\left(\hat{1}\left(z_{P_i}\right),2,\ldots,-\hat{P}_i^{-h}\left(z_{P_i}\right)\right)\frac{i}{P_i^2}A_{n+2-i}\left(\hat{P}_i^{h}\left(z_{P_i}\right),i,\ldots,n-1,\hat{n}\left(z_{P_i}\right)\right)$$

[Zoia, 2021]



Motivation

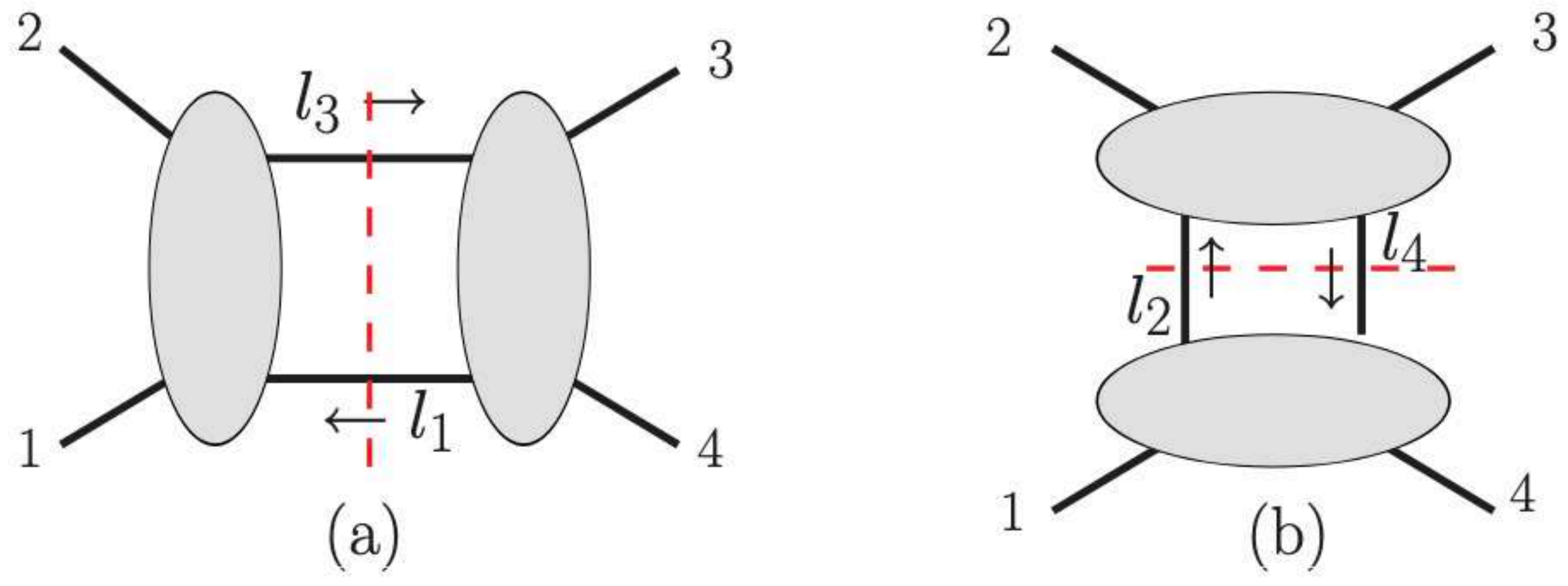
2. Analytic motivation

Loop-level

Loop level amplitudes involve integration of internal (virtual) momenta

Generalised unitarity methods allow us to construct one-loop amplitudes from tree-level amplitudes.

$$C_s = \sum_{\text{states}} A^{\text{tree}}(-l_1, 1, 2, l_3) A^{\text{tree}}(-l_3, 3, 4, l_1)$$

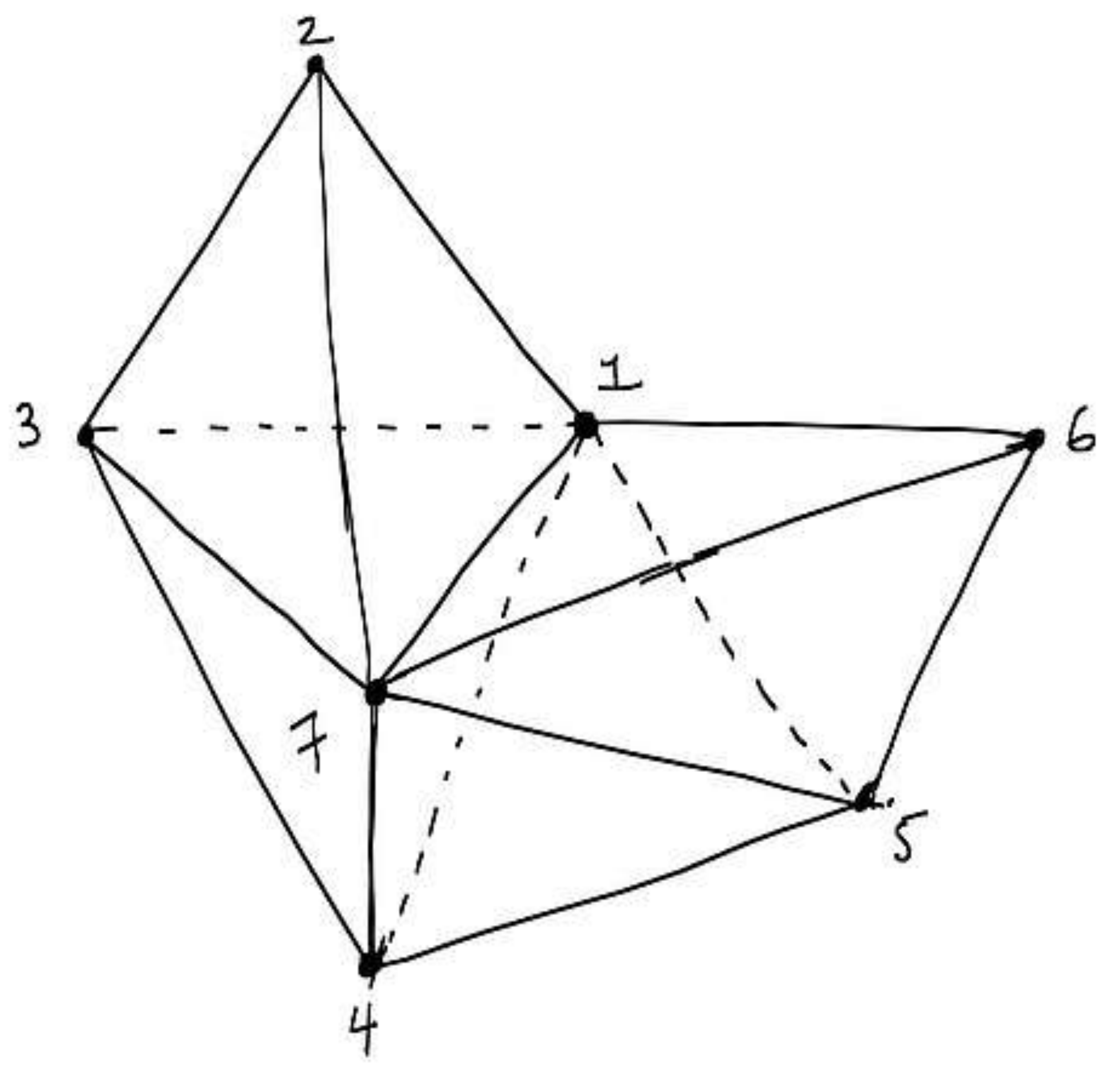


Motivation

2. Analytic motivation

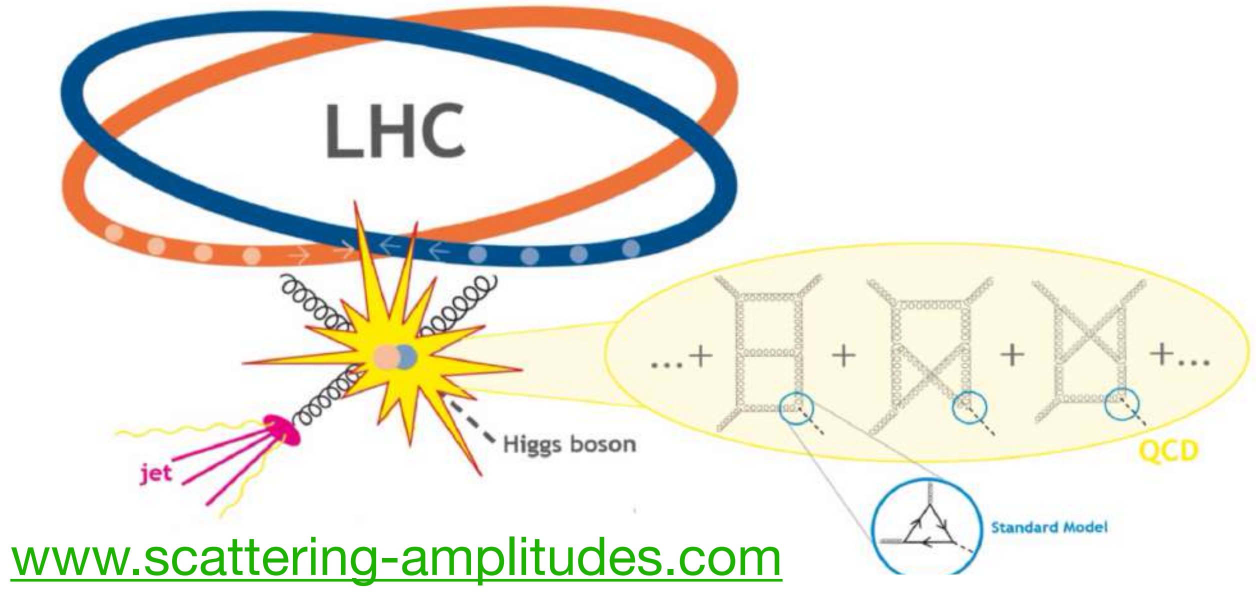
Positive geometry, cluster algebra, duality…

Amplituhedron

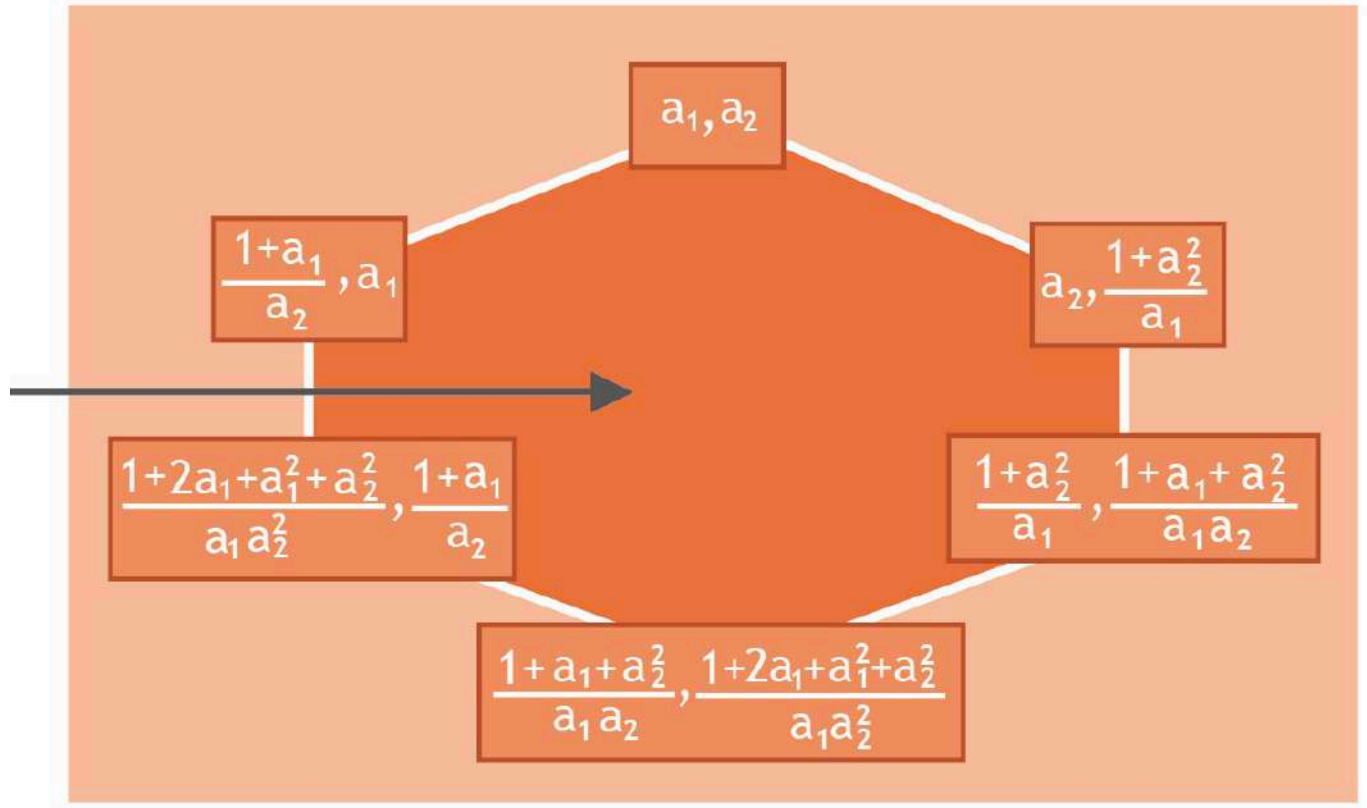


[Arkani-Hamed, Trnka, 2013]

Cluster algebra



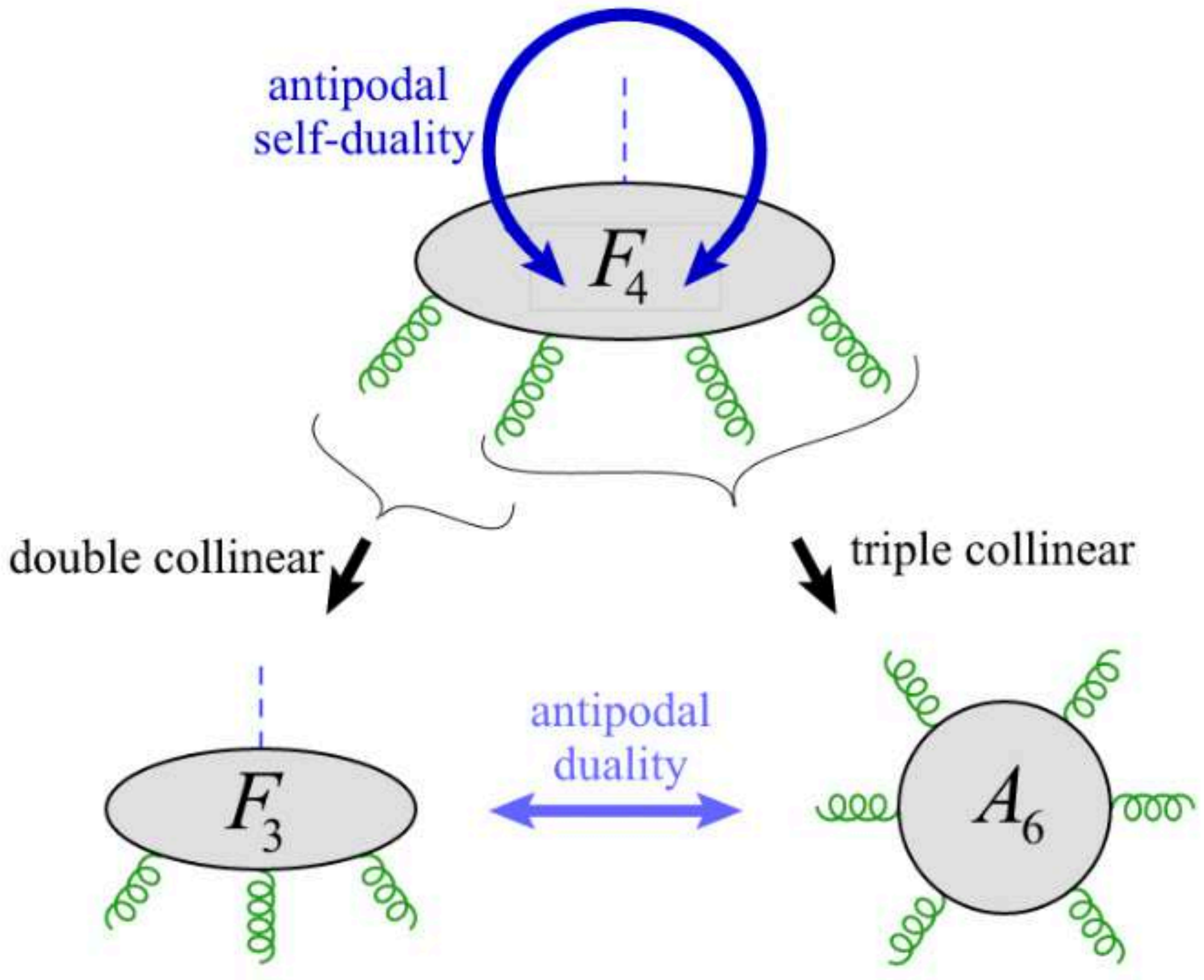
www.scattering-amplitudes.com



[Chicherin, Henn, Papathanasiou 2020]

Antipodal duality

[Dixon, Gürdoğan, Liu, McLeod, Wihlem 2022]



[Dixon, Gürdoğan, McLeod, Wihlem 2021]

Analytic computation of Feynman integrals

Although we have the integrand expressions, most of time we still need to perform the integration to obtain the amplitudes

Q. Why analytic integration?

1. Time efficiency

Numerics

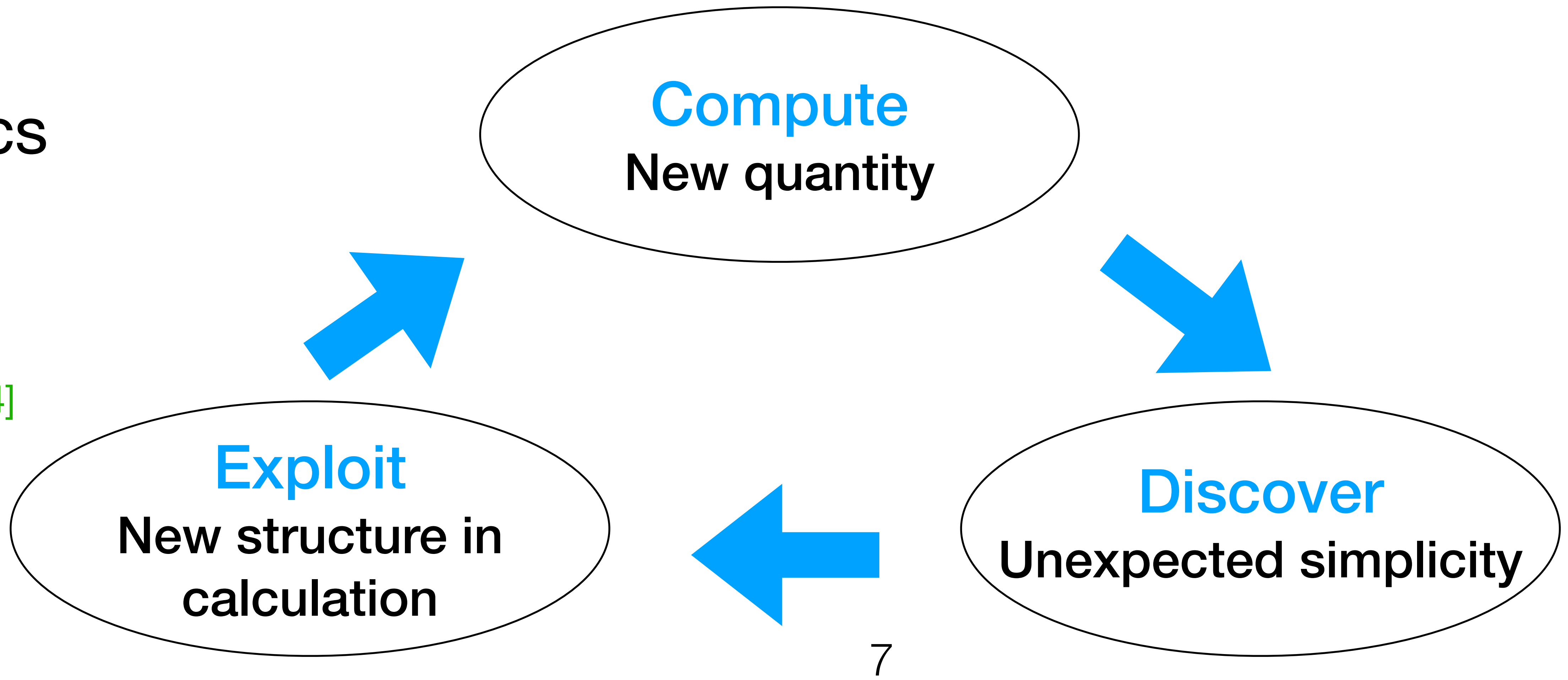
Family	AMFlow	One–fold integration
Double–box (db)	$\sim$ 3.1 h	$\sim$ 7.5 min
Pentagon–trinagle (pt)	$\sim$ 45 min	$\sim$ 1.8 min
Hexagon–bubble (hb)	$\sim$ 25 min	$\sim$ 1.1 min

[Henn, et al 2024]

2. Physics, Mathematics

The "Virtuous Cycle"

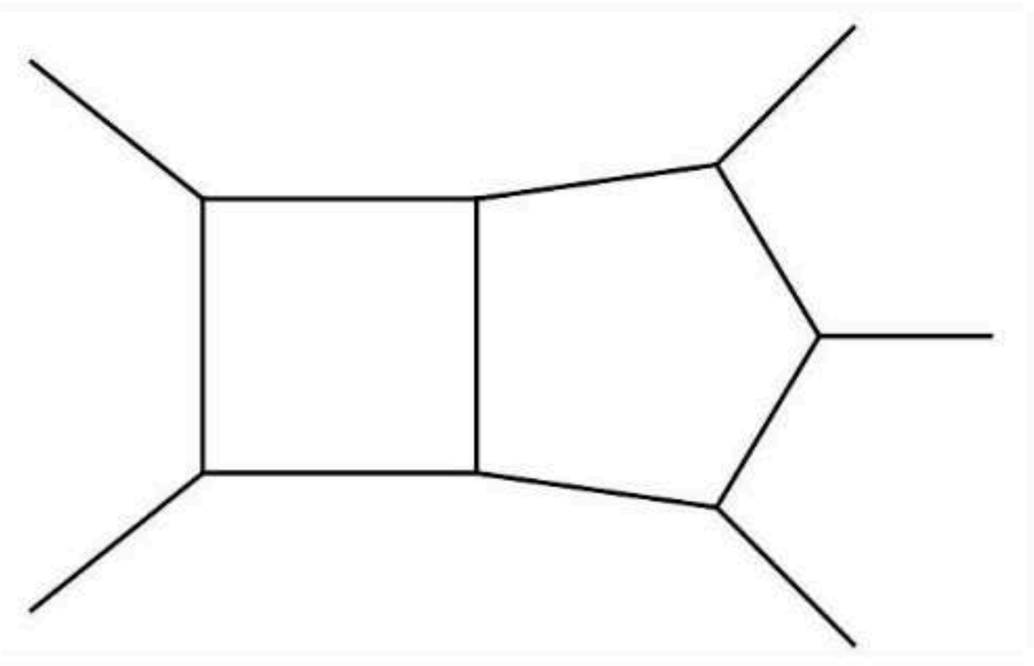
[Bern's talk in Snowmass 2022]
[Mcleod's talk in Galaxy meets QCD 2024]



Analytic computation of Feynman integrals

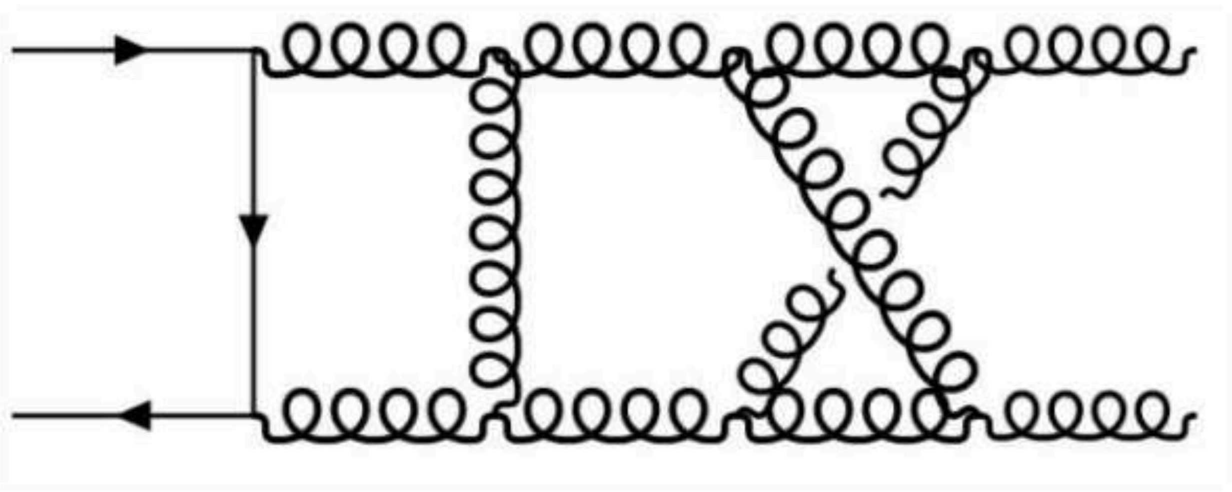
Although we have the integrand expressions, we still need to perform the integration to obtain the amplitudes

- State-of-the-art computations



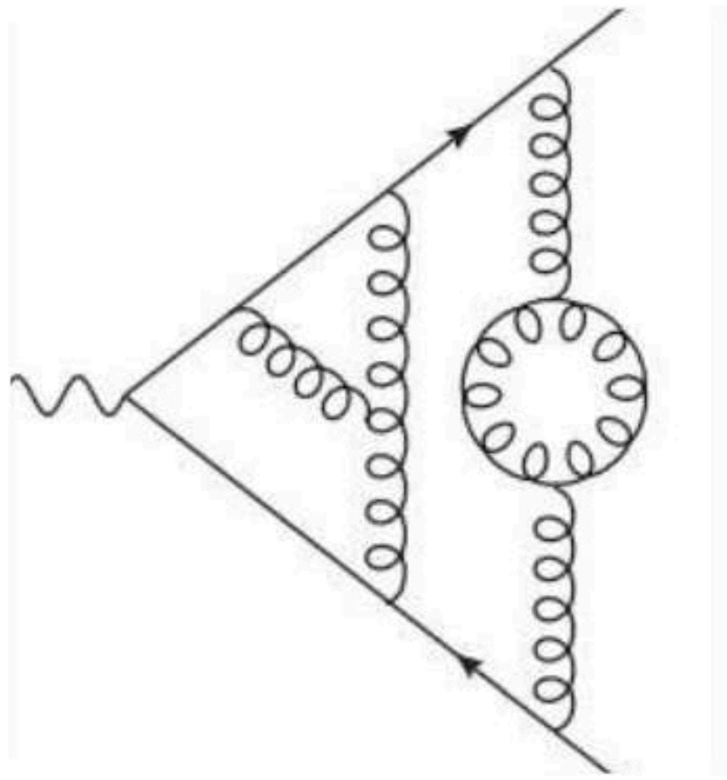
2 loop 5point

Abreu, Agarwal, Badger, Buccioni, Chawdhry, Chicherin, Czakon, de Laurentis, Febres-Cordero, Gambuti, Gehrmann, Henn, Ita, Lo Presti, Manteuffel, Ma, Mitov, Page, Peraro, Pochelet, Schabinger, Sotnikov, Tancredi, Zhang, …



3 loop 4 point

Bargiela, Bobadilla, Canko, Caola, Jakubcik, Gambuti, Gehrmann, Henn, Lim, Mella, Mistlberger, Wasser, Manteuffel, Syrrakos, Smirnov, Tancredi, …



4 loop 3 point

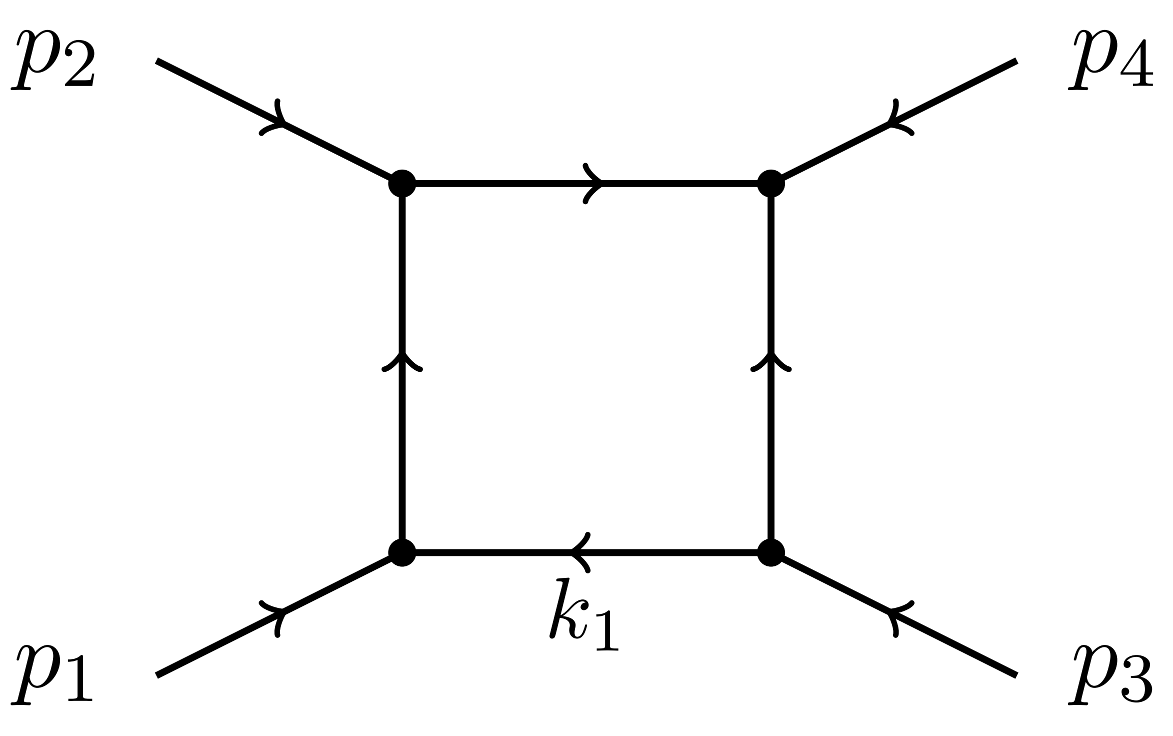
Henn, Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Stainhauser,…

Differential equation method

Integration-by-parts (IBP) relations

[Chetyrkin, Tkachov 1981]

Ex) 1loop all on-shell



$$G_{a_1,a_2,a_3,a_4} = \int \frac{dk}{i\pi^2} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}}$$

Integral family

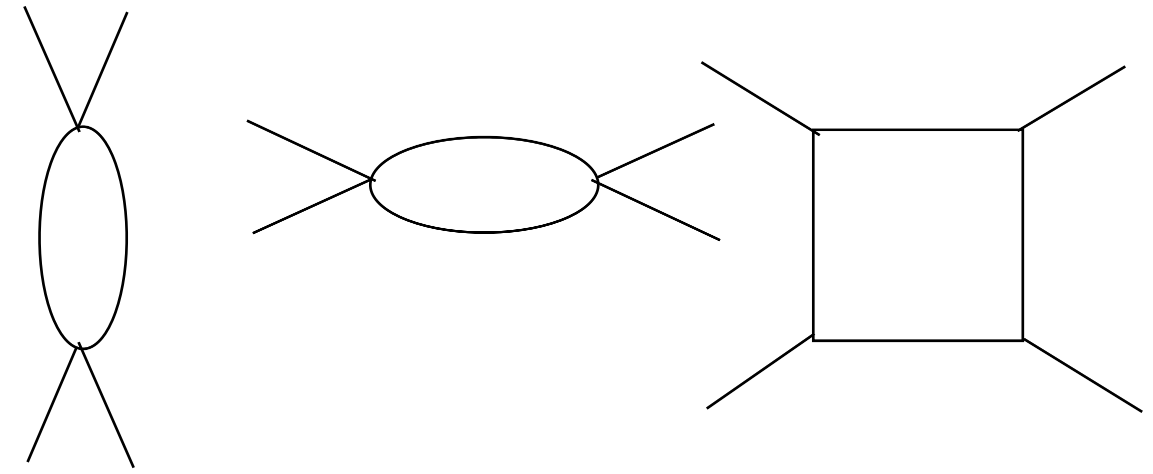
Total derivatives vanish in dimensional regularization.

$$0 = \int \frac{d^D y}{i\pi^{D/2}} \frac{\partial}{\partial y^\mu} \xi^\mu \prod_{i=1}^{4} \frac{1}{\left[-(y-y_i)^2\right]^{a_i}}$$

$$G_{2,1,1,1} = \frac{D-5}{s} G_{1,1,1,1} - \frac{4\left(D-5\right)(D-3)}{\left(D-6\right) st^2} G_{0,1,0,1}$$

$$G_{1,1,0,1} = \frac{2\,(D-3)}{(D-4)\,s} G_{0,1,0,1}$$

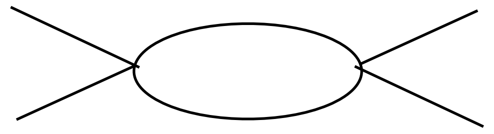
Basis of the box integral : $\vec{f} = \{G_{0,1,0,1}, G_{1,0,1,0}, G_{1,1,1,1}\}$ ⟶ Master integral



Differential equation method

Constructing differential equations

$$\partial_s = (a_1 p_1 + a_2 p_2 + a_3 p_3) \cdot \partial_{p_1} + (b_1 p_1 + b_2 p_2 + b_3 p_3) \cdot \partial_{p_2}$$

Ex) 1loop all on-shell 

$$\partial_s G_{1,0,1,0} = \frac{1}{2s} G_{0,0,2,0} + \frac{1}{2t} G_{1,-1,2,0} - \frac{1}{2s} G_{1,0,1,0} - \frac{1}{2t} G_{1,0,2,-1} + \frac{1}{2} G_{1,0,2,0}$$

IBP reduction

$$\frac{\partial \vec{f}}{\partial s} = A_s\left(\{s,t\},\epsilon\right) \cdot \vec{f}$$

$$\frac{\partial \vec{f}}{\partial t} = A_t\left(\{s,t\},\epsilon\right) \cdot \vec{f}$$

$$A_s = \begin{pmatrix} \frac{1-\epsilon}{s} & 0 & 0 \\ 0 & -\frac{1}{2s} + \frac{1}{2(s+t)} & 0 \\ -\frac{2(-1+2\epsilon)}{s^2(s+t)} & \frac{2(-1+2\epsilon)}{s^2 t} - \frac{2(-1+2\epsilon)}{s^2(s+t)} & -\frac{1+2\epsilon}{2s} + \frac{1+2\epsilon}{2(s+t)} \end{pmatrix}$$

Differential equation method

Canonical differential equations

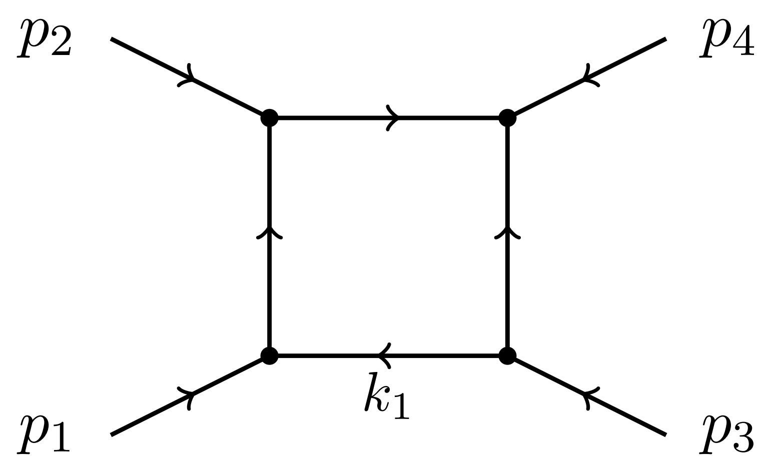
[Henn 2013]

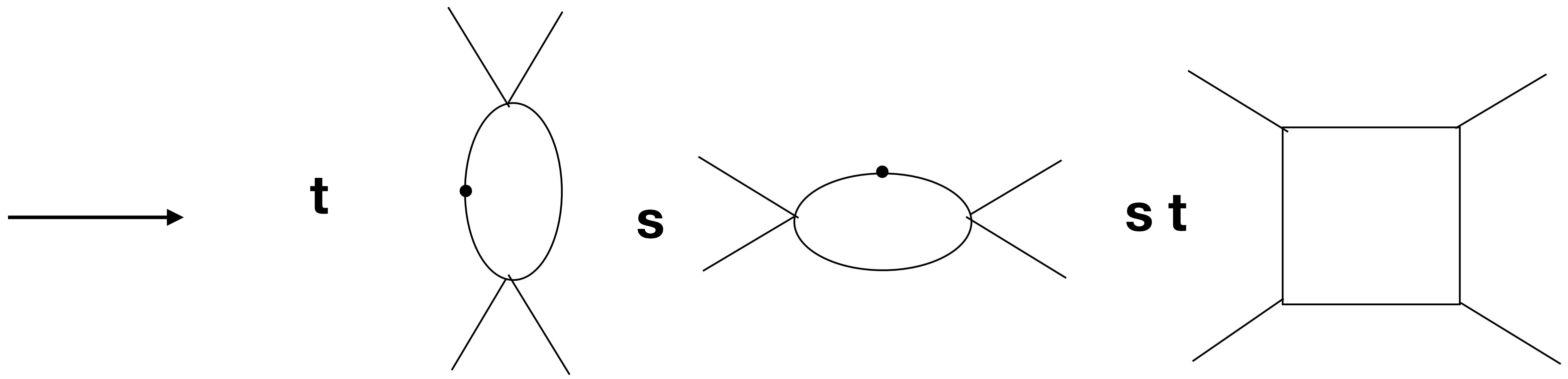
Good choice of basis for Feynman integrals can significantly simplify the computation of differential equation.

$$d\vec{f}\left(\vec{x},\epsilon\right) = \epsilon\left(d\tilde{A}\right)\vec{f}\left(\vec{x};\epsilon\right), \quad \text{with } \tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right]$$

Uniform transcendental weight (UT) integral $\vec{f}(x,\epsilon) = \sum_{k\geq 0} \epsilon^k \vec{f}^{(k)}(x)$

Ex) 1loop all on-shell





$$A_s = \begin{pmatrix} \frac{\epsilon}{s} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\epsilon}{s} - \frac{2\epsilon}{s+t} & -\frac{2\epsilon}{s+t} & -\frac{\epsilon}{s} + \frac{\epsilon}{s+t} \end{pmatrix}$$

$\{G_{0,1,0,1}, G_{1,0,1,0}, G_{1,1,1,1}\}$

Canonical basis of the box integral : $\{\frac{t}{\epsilon}G_{0,1,0,2}, \frac{s}{\epsilon}G_{1,0,2,0}, stG_{1,1,1,1}\}$

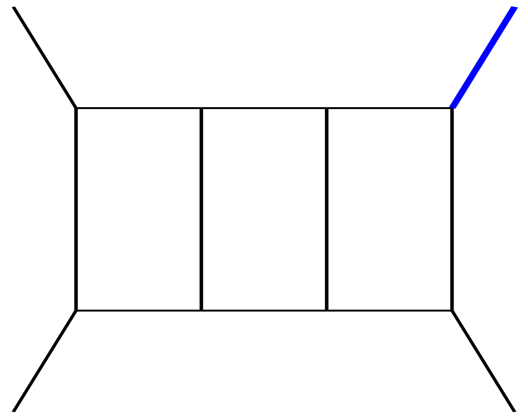
Differential equation method

Canonical differential equations

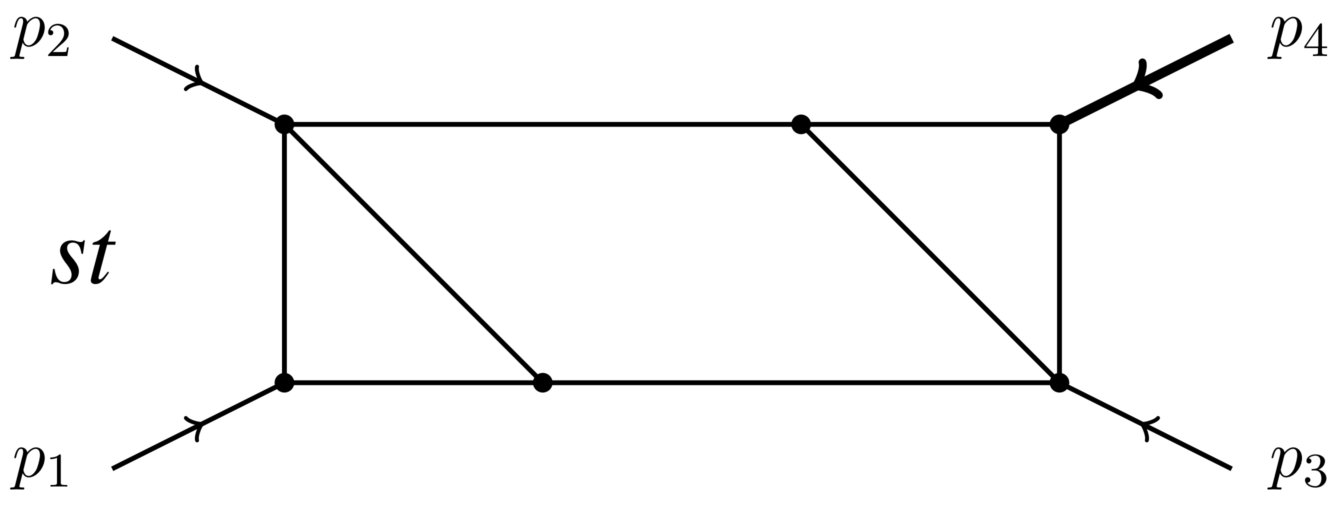
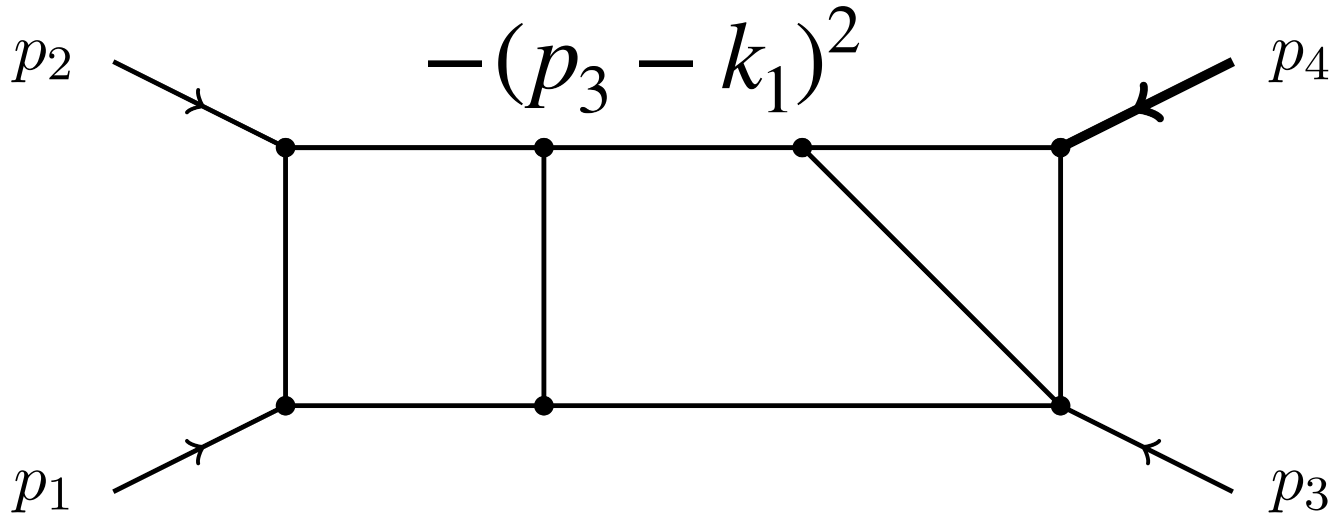
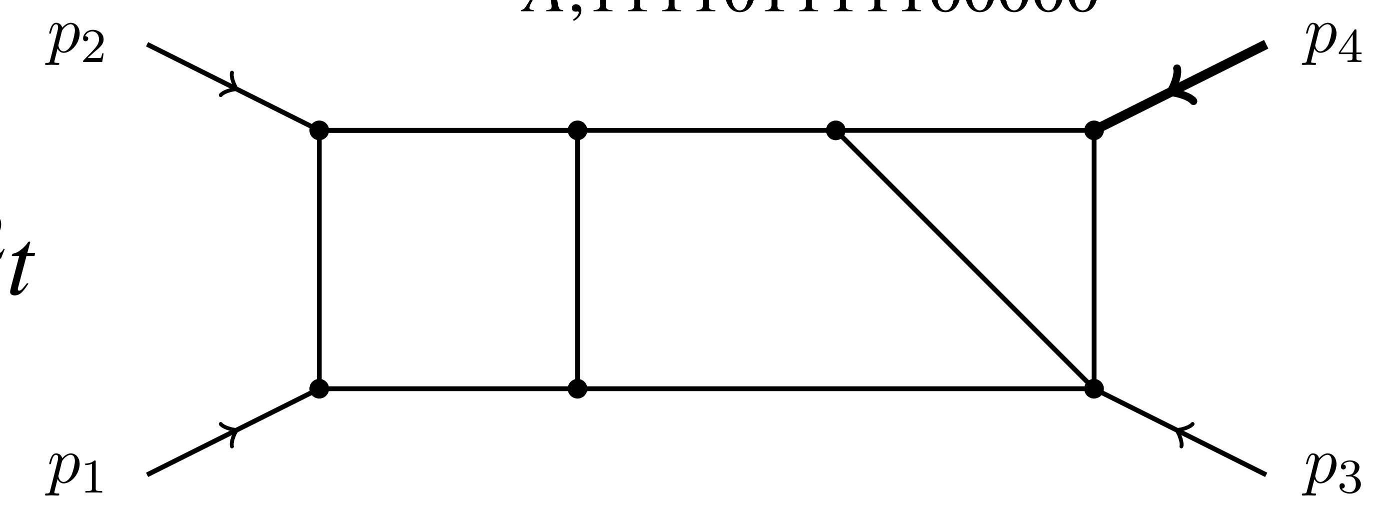
Dlog-integral

Feynman integrands that can be written as $d\log$-form (conjecturally) evaluate to **uniform weight** functions after integration.

An integrand admitting a $d\log$-form can be written as $\mathscr{I} = \sum_k c_k d\log g_1^{(k)} \wedge d\log g_2^{(k)} \wedge \ldots \wedge d\log g_n^{(k)}$

Ex) Family A1 

$stJ_{A;101101111100000}$ $\quad +s^2 J_{A;1111011111-10000}$ $\quad +s^2 t J_{A;111101111100000}$

st  $+s^2$  $+s^2 t$ 

[Wasser 2020]

Iterated integral

Solution of differential equations : $\vec{f}\left(\vec{x},\epsilon\right) = \mathbb{P}\exp\left[\epsilon\int_{\gamma} d\tilde{A}\right]\vec{f}_0(\epsilon)$

where $\vec{f}_0(\epsilon)$ is a boundary vector

$\gamma * \left(\omega_i\right) = k_i(t)\,dt$, function k_i are defined by pulling back the 1-form ω_i to the interval [0,1]

$$\vec{f}^{(0)} = \vec{f}_0^{(0)}$$

$$\vec{f}^{(1)}(z_1, z_2) = M_{z_1 z_2}^{(1)}\vec{f}_0^{(0)} + \vec{f}_0^{(1)}$$

$$\vec{f}^{(2)}(z_1, z_2) = M_{z_1 z_2}^{(2)}\vec{f}_0^{(0)} + M_{z_1 z_2}^{(1)}\vec{f}_0^{(1)} + \vec{f}_0^{(2)}$$

$$\ldots$$

$$\vec{f}^{(6)}(z_1, z_2) = M_{z_1 z_2}^{(6)}\vec{f}_0^{(0)} + M_{z_1 z_2}^{(5)}\vec{f}_0^{(1)} + \ldots + M_{z_1 z_2}^{(1)}\vec{f}_0^{(5)} + \vec{f}_0^{(6)}$$

Iterated integrals

Q. How does $f^{(n)}$ look like?

A. Iterated integrals

An ordinary line integral is given by $\int_\gamma \omega_1 = \int_{[0,1]} \gamma^*\left(\omega_1\right) = \int_0^1 k_1\left(t_1\right) dt_1$

Iterated integral of $\omega_1 \dots \omega_n$ along γ is defined by $\int_\gamma \omega_1 \dots \omega_n = \int_{0 \le t_1 \le \dots \le t_n \le 1} k_1\left(t_1\right) dt_1 \dots k_n\left(t_n\right) dt_n$

[Chen 1977]

Ex) Polylogarithms

$$\log(x) = \int_0^x \frac{dt}{t}, \qquad \mathrm{Li}_1(x) = -\log(1-x) = \int_0^x \frac{dt}{1-t}, \qquad \mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t).$$

If the alphabet is rational functions , one can write the answer in terms of Goncharov polylogarithms

[Goncharov 2001]

$$G(\vec{a}_n; z) \equiv G(\vec{a}_1, \vec{a}_{n-1}; z) \equiv \int_0^z \frac{dt}{t-a_1} G(\vec{a}_{n-1}; t)$$

with $G\left(a_1; z\right) = \int_0^z \frac{dt}{t-a_1}$ and $G(\vec{0}_n; z) \equiv \frac{1}{n!} \log^n(z)$

Function space of Feynman integrals and scattering amplitudes

Transcendental function : $I\left(\omega_1, \ldots, \omega_n; \vec{x}\right) = \int_\gamma \omega_1 \omega_2 \cdots \omega_n,$ $\quad I\left(; \vec{x}\right) = 1,$

Same definition can be extended to transcendental number $\xi_n = \pi^2, \zeta_n, \ldots,$ which correspond to special values of the iterated integrals, and we also assign weight −1 to ϵ.

The basis of our space is $b_{\mathcal{T}_\omega} = \{\epsilon^{-a} \xi_n^b I\left(\omega_1, \ldots, \omega_c; \vec{x}\right)\}$, with weight $w = a + nb + c$

Ex) $st^2 J_{E1;1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$

$$= \frac{2}{9} + \epsilon\left(-\frac{2}{3}I\left(\omega_1\right) - \frac{2}{3}I\left(\omega_2\right)\right) + \epsilon^2\left(\frac{8\pi^2}{27} - \frac{2}{3}I\left(\omega_1, \omega_4\right) + 2I\left(\omega_1, \omega_1\right) + 2I\left(\omega_1, \omega_2\right) + 2I\left(\omega_2, \omega_1\right) - \frac{2}{3}I\left(\omega_2, \omega_5\right) + 2I\left(\omega_2, \omega_2\right)\right) + \mathcal{O}\left(\epsilon^3\right)$$

$$\omega_1 = \frac{-s}{-m^2}, \omega_2 = \frac{-t}{-m^2}$$

Function space of Feynman integrals and scattering amplitudes

The basis of our space is $b_{\mathcal{T}_\omega} = \{\epsilon^{-a}\xi_n^b I\left(\omega_1, \ldots, \omega_c; \vec{x}\right)\}$, with weight $w = a + nb + c$

Ex) $st^2 J_{E1;1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$

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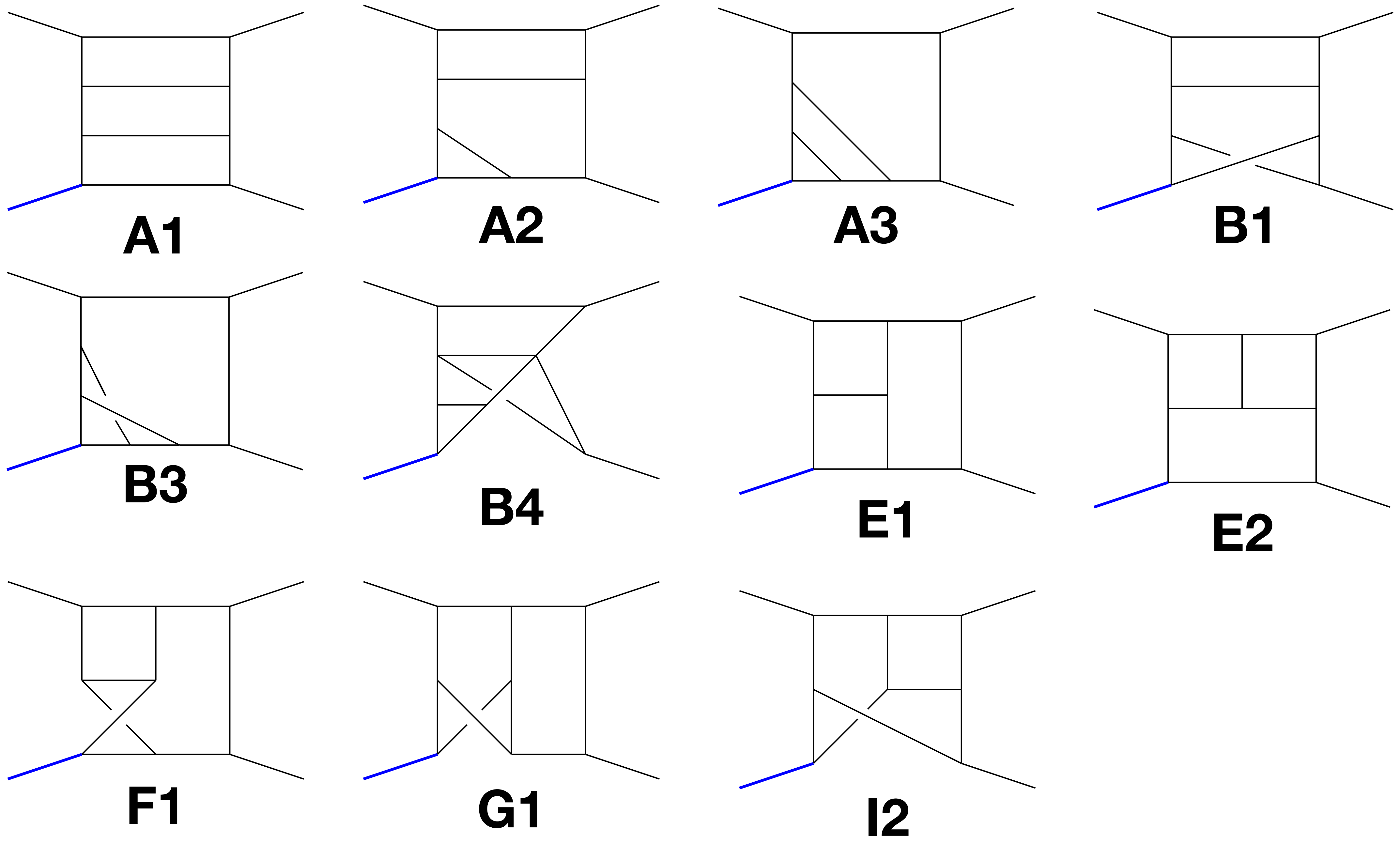
$$\omega_1 = \frac{-s}{-m^2}, \omega_2 = \frac{-t}{-m^2}$$

Important information

1. What kinds of ω_i can we have? ⟶ (Alphabet) letter
2. What kinds of sequences of $\{\omega_i\}$ can we have? ⟶ Symbol

Function space of Feynman integrals and scattering amplitudes

Example. Three-loop N=4SYM three-point trϕ^2 form factor (Leading color)



$$\vec{\alpha} = \{p_4^2, s, t, p_4^2 - s - t, p_4^2 - s, p_4^2 - t, s + t, \frac{(p_4^2 - s - t)s - R}{(p_4^2 - s - t)s + R}, \frac{st - R}{st + R}, p_4^4 - t\left(p_4^2 + s\right), p_4^4 - s\left(p_4^2 + t\right),$$

with $R = \sqrt{-p_4^2 s(p_4^2 - s - t)t}$

➡ **20 letters (including kinematic crossings)**

	# MI
A1	83
A2	100
A3	80
B1	150
B3	90
B4	143
E1	166
E2	117
F1	214
G1	254
I2	305

(Family : reducible top sector)

Function space of Feynman integrals and scattering amplitudes

Example. Three-loop N=4SYM three-point trϕ^2 form factor (Leading color)

- **Function space**

  $\vec{\alpha} = \{u, v, w, 1-u, 1-v, 1-w\}$, with $u = \frac{-s}{-p_4^2}, v = \frac{-t}{-p_4^2}, w = 1-u-v.$

  Quadratic letters and square root letters are cancelled out! $\longrightarrow$ C_2 cluster algebra space

- **Adjacency conditions**

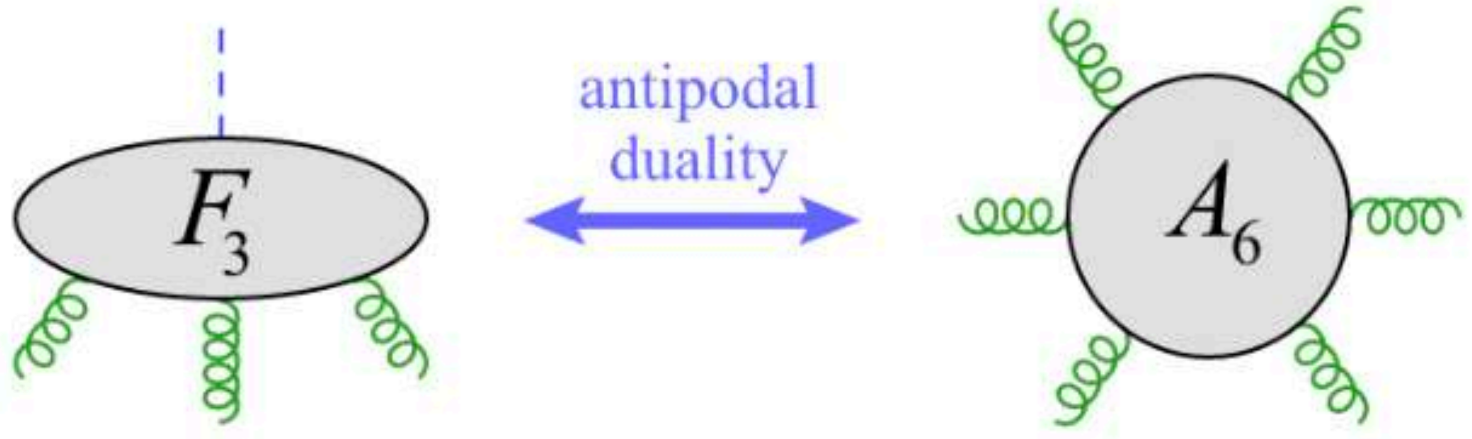
  $$a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w}.$$

  1. ~~$\ldots d \otimes e \ldots$~~, ~~$\ldots e \otimes f \ldots$~~, ~~$\ldots f \otimes d \ldots$~~ + swap
  2. ~~$\ldots a \otimes d \ldots$~~, ~~$\ldots b \otimes e \ldots$~~, ~~$\ldots c \otimes f \ldots$~~ + swap
  3. ~~$\ldots a \otimes abc \otimes b \ldots$~~ + dihedral

Function space of Feynman integrals and scattering amplitudes

Example. Three-loop N=4SYM three-point trϕ^2 form factor (Leading color)

- **Antipodal duality**



[Dixon, Gürdoğan, McLeod, Wihlem 2021]

$$\alpha_{F_3} = \{u, v, w, 1-u, 1-v, 1-w\}, \quad \text{with} \quad u = \frac{s_{12}}{s_{123}}, v = \frac{s_{23}}{s_{123}}, w = \frac{s_{13}}{s_{123}},$$

$$\alpha_{A_6} = \{\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\},$$

$$\text{with} \quad \hat{u} = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \hat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \quad \hat{y}_u = \frac{\hat{u}-z_+}{\hat{u}-z_-}, \hat{y}_v = \frac{\hat{v}-z_+}{\hat{v}-z_-}, \hat{y}_w = \frac{\hat{w}-z_+}{\hat{w}-z_-},$$

$$\text{where} \quad z_\pm = \frac{1}{2}\left[-1+\hat{u}+\hat{v}+\hat{w}\pm\sqrt{\Delta}\right],$$

Antipodal map : $$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)\Big|_{\hat{u}_i \to \hat{u}_i(u,v,w)}$$

$$\hat{u}(u,v,w) = \frac{vw}{(1-v)(1-w)},$$

$$\hat{v}(u,v,w) = \frac{uw}{(1-u)(1-w)},$$

$$\hat{w}(u,v,w) = \frac{uv}{(1-u)(1-v)}.$$

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

→ **Bootstrapping amplitude up to 8 loops**

[Dixon, Liu, 2023]

Twisted Feynman integrals

Twisted Feynman integrals

Deformations of typical Feynman integrals where the integrand is modified by an exponential factor.

$$I_L = \int_{l_1,\ldots,l_L} X(l_1,\cdots,l_L) \to \qquad I_L\left[\alpha_1^\mu,\cdots,\alpha_L^\mu\right] = \int_{l_1,\cdots,l_L} X(l_1,\cdots,l_L)e^{i\sum_{n=0}^{L} l_n\cdot\alpha_n}$$

Ex) generating functional for tensor integrals, spin-resumed dynamics of black holes, dipole scattering

Analytic structure of twisted Feynman integrals

Singularity structures

For usual Feynman integrals, the singularity structures are relatively well understood

Landau singularities

$$I(p_i) = \int_0^1 \prod_{j=1}^{N} da_j \delta\left(\sum_{i=1}^{N} a_i - 1\right) \int \prod_{l=1}^{L} \frac{d^d k_l}{(2\pi)^d} \frac{\mathcal{N}(a_j, k_l, p_i)}{\left[\mathcal{D}(a_j, k_l, p_i)\right]^N}, \xrightarrow{\text{Feynman tricks}} I = \frac{1}{\prod_{i=1}^{n} \Gamma(\nu_i)} \int_{\alpha_i \geq 0} d^n a \delta\left(1 - \sum_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} a_i^{\nu_i - 1}\right) \frac{\mathcal{U}(a)^{\nu - \frac{(l+1)D}{2}}}{\mathcal{F}(a)^{\nu - \frac{lD}{2}}} \int_0^\infty dt t^{\nu - \frac{lD}{2} - 1} e^{-t},$$

$$D(a_j, k_l, p_i) \equiv \sum_{j=1}^{N} a_j (q_j^2 - m_j^2) + i\eta = 0,$$

$$\frac{\partial}{\partial k_j^\mu} \mathcal{D}(a_j, k_l, p_i) = 0$$

$a_i = 0$, for $i \in S$,

$\frac{\partial \mathcal{F}}{\partial a_j} = 0$ for $j = \{1, \ldots, n\} \backslash S$

Alphabet letters $\{\omega_i\} \in$ Leading Singularities $\sim$ Landau singularities

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For twisted Feynman integrals, the Feynman parametrisation does not give only polynomials

$$I = \frac{1}{\prod_{i=1}^{n}\Gamma(\nu_i)}\int_{\alpha_i\geq 0} d^n a\delta\left(1-\sum_{i=1}^{n}a_i\right)\left(\prod_{i=1}^{n}a_i^{\nu_i-1}\right)\frac{1}{\mathcal{U}^{\frac{D}{2}}}\left(\frac{\mathcal{F}_2}{\mathcal{F}_0}\right)^{\frac{2\nu-lD}{4}} e^{-\mathcal{P}_1}K_{\nu-\frac{lD}{2}}\left(2\tilde{\mathcal{P}}\right),$$

Alphabet letters $\{\omega_i\} \notin$ Leading Singularities

⟶ How to apply the same techniques to Feynman integrals?

e.g. Exponential periods?

Conclusion

- Scattering amplitudes are crucial for precision computations of high energy physics observables
- It also tells us how the physics are reflected on those objects and gives us new way of viewing the physics
- Computing Feynman integrals are crucial in amplitudes and its analytic structures are key to push the limit forward
- Differential equation method is the cutting edge technique for Feynman integrals
- Scattering amplitudes have richer analytic properties than Feynman integrals and they can be used for bootstrapping approach
- Generalized Feynman integrals are appearing also in different contexts of physics while its analytic structure is not fully understood

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Thank You!