

# Faithful realizations of semi-classical truncations for effective quantum dynamics

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# Outline

- 1 Introduction
- 2 Canonical effective methods
- 3 Algebra of semiclassical truncations
- 4 Faithful realization of semiclassical truncations
- 5 Toy model: Coupled oscillators
- 6 Summary and applications

# Motivation

- **Semi-classical physics** is useful for systems with quantum corrections, instead of solving the full quantum theory, e.g. low-energy effective action, time-dependent variational principle, **canonical effective methods**
- **Canonical effective methods**: Parametrize a state by expectation values and moments:  $\langle \hat{q} \rangle, \langle \hat{\pi} \rangle, \Delta(q^a \pi^b)$  and approximate quantum dynamics by coupling  $\langle \hat{q} \rangle, \langle \hat{\pi} \rangle$  to  $\Delta(q^a \pi^b)$  [Bojowald, Prezhdo, etc.]

## Advantages & Challenge

- i. Need only expectation-value functional on the algebra
- ii. PDE for wavefunctions  $\rightarrow$  system of coupled ODEs
- iii. However, moments of a state **do not** form canonically conjugate pairs (mostly non-linear)

## Goal

Construct **canonical realizations** of Poisson systems (Casimir-Darboux coordinates) for semiclassical truncations.

# Setup 1 - Kinematical setting

- Canonical (unital) algebra  $\mathcal{A}$  generated by self-adjoint  $Q_j, \Pi_k, 1 \leq j, k \leq N$ , with canonical commutation relations:

$$[Q_j, \Pi_k] = i\hbar\delta_{jk}$$

- States: positive linear functionals  $\omega : \mathcal{A} \rightarrow \mathbb{C}$
- Basic expectation values:  $q_j = \langle \hat{Q}_j \rangle \in \mathbb{R}$  and  $\pi_k = \langle \hat{\Pi}_k \rangle \in \mathbb{R}$

## Definition 1

$$\Delta(q_1^{k_1} \cdots q_N^{k_N} \pi_1^{l_1} \cdots \pi_N^{l_N}) = \langle [(\hat{Q}_1 - q_1)^{k_1} \cdots (\hat{Q}_N - q_N)^{k_N} (\hat{\Pi}_1 - \pi_1)^{l_1} \cdots (\hat{\Pi}_N - \pi_N)^{l_N}]_{\text{Weyl}} \rangle$$

- Note:  $\langle \hat{a} \rangle$  to denote  $\omega(a)$  with  $\langle \hat{a} - a \rangle = 0$  for any operator  $\hat{a}$  and  $\sum_{i=1}^N (k_i + l_i) \geq 2, \forall i, k_i, l_i \in \mathbb{N}$

## Poisson structure

$\{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} := \frac{1}{i\hbar} \langle [\hat{A}, \hat{B}] \rangle$  (Extended to all moments via linearity and Leibniz rule)

e.g.  $\{q_j, \pi_k\} = \delta_{jk}, \{q_j, \Delta\} = \{\pi_k, \Delta\} = 0$  ( $\{q, \Delta_n\} \propto \sum \langle \hat{Q} - q \rangle^a \langle \hat{\Pi} - \pi \rangle^{n-a-1}$ )

## Setup 2 - Semiclassical truncation

### Definition 2

A state on  $\mathcal{A}$  is *semiclassical* if its moments obey the hierarchy:

$$\Delta(q_1^{k_1} \cdots \pi_N^{l_N}) = \mathcal{O}\left(\hbar^{\frac{1}{2} \sum_n (l_n + k_n)}\right)$$

### Definition 3

The *semiclassical truncation of order  $o \geq 2$*  of a quantum system with  $\mathcal{A}$  is a finite-dimensional manifold  $\mathcal{P}_o$  with coordinates  $q_j, \pi_k$  and all moments such that  $\sum_n (l_n + k_n) \leq o$

### Boundary components from Cauchy-Schwarz inequality

$o = 2$  :  $\Delta(q_j^2)\Delta(\pi_k^2) - \Delta(q_j\pi_k)^2 \geq \frac{\hbar^2}{4} \delta_{jk}$  (Higher-order versions exist for  $o > 2$ )

### Example ( $N = 1$ and $o = 2$ )

$$\begin{aligned} \{q, \pi\} &= 1, & \{\Delta(q^2), \Delta(q\pi)\} &= 2\Delta(q^2), \\ \{\Delta(q\pi), \Delta(\pi^2)\} &= 2\Delta(\pi^2), & \{\Delta(q^2), \Delta(\pi^2)\} &= 4\Delta(q\pi) \end{aligned}$$

## Setup 3 - Dynamics

### Quantum Hamiltonian

$$H_Q(\langle \cdot \rangle, \Delta) = \langle \hat{H} \rangle_{\langle \cdot \rangle, \Delta} := \langle H(q_j + (\hat{Q}_j - q_j), \pi_k + (\hat{\Pi}_k - \pi_k)) \rangle$$

### Effective Hamiltonian of order $o$

$$H_{\text{eff},o} := H_Q|_{\sum_n(l_n+k_n)\leq o} = H(q_j, \pi_k) + \sum_{\sum_n(l_n+k_n)=2}^o \frac{\partial^n H(q_j, \pi_k)}{\partial q_1^{l_1} \dots \partial q_N^{l_N} \partial \pi_1^{k_1} \dots \partial \pi_N^{k_N}} \frac{\Delta(q_1^{l_1} \dots q_N^{l_N} \pi_1^{k_1} \dots \pi_N^{k_N})}{l_1! \dots l_N! k_1! \dots k_N!}$$

- A formal Taylor expansion in  $\hat{Q}_j - q_j$  and  $\hat{\Pi}_k - \pi_k$  and  $H(q, \pi)$  is the classical Hamiltonian ( $H \in \mathcal{A}$ )
- Polynomial Hamiltonians are of particular interest.

### Effective equations of motion

$\dot{f}(q, \pi, \Delta) = \{f(\langle \cdot \rangle, \Delta), H_{\text{eff},o}\}$  are truncations of Heisenberg's equations of motion evaluated in a state

# Setup 4 - Connection to QM and QFT

## Number of moments $\Delta$

$$\#_{\Delta} = \frac{1+o}{2N} \binom{2N+o}{o+1} - (2N+1)$$

$\mathcal{O}(\hbar^{o/2})/N\text{-DOF}$	1	2	3	...	$\infty$
$\hbar$	3	10	21		$\infty$
$\hbar^{3/2}$	7	30	77		$\infty$
$\hbar^2$	12	65	203		$\infty$
...					...
$\hbar^{\infty}$	$\infty$ (QM)	$\infty$	$\infty$	...	$\infty$ (QFT)

**Table 1:** Number of moments for a given semi-classical order and number of classical degrees of freedom

# General Poisson Bracket Formula for $N = 1$

- [Bojowald-Skirzewski 2006]:

$$\{\Delta(q^b \pi^a), \Delta(q^d \pi^c)\} = ad\Delta(q^b \pi^{a-1})\Delta(q^{d-1} \pi^c) - bc\Delta(q^{b-1} \pi^a)\Delta(q^d \pi^{c-1}) \\ + \sum_{\text{odd } n=1}^M \left(\frac{i\hbar}{2}\right)^{n-1} K_{abcd}^n \Delta(q^{b+d-n} \pi^{a+c-n})$$

with  $M = \min(a + c, b + d, a + b, c + d)$  and

$$K_{abcd}^n = \sum_{m=0}^n (-1)^m m!(n-m)! \binom{a}{m} \binom{b}{n-m} \binom{c}{n-m} \binom{d}{m}$$

- Brackets are **linear** in moments if:  $a + b = 2$  or  $c + d = 2$  or  $o = 3$

## Why?

The nonlinear terms are products of moments:

- $\{\Delta_n, \Delta_m\} \propto \Delta_{n-1} \Delta_{m-1} + \dots$
- For  $n, m \geq 3$ :  $n + m - 2 \geq 4$  (For  $n, m = 2$ :  $\Delta_{n-1}, \Delta_{m-1} \mapsto 0$ )
- Contribute at fourth order and higher

# Poisson structure of second-order ( $\sigma = 2$ ) truncation

For  $N$  degrees of freedom, let  $x_i = (q_1, \dots, q_N, \pi_1, \dots, \pi_N)$ .

Poisson brackets  $\text{ad}(\Delta)(\Delta') = \{\Delta, \Delta'\}$

$$\{\Delta(x_i x_j), \Delta(x_k x_l)\} = \sum_{m \leq n} f_{ij;kl}^{mn} \Delta(x_m x_n) = \sum_{m,n} f_{ij;kl}^{(mn)} \Delta(x_m x_n)$$

Structure constants:

$$f_{ij;kl}^{mn} = \tau_{ik} \delta_j^m \delta_l^n + \tau_{il} \delta_j^m \delta_k^n + \tau_{jk} \delta_i^m \delta_l^n + \tau_{jl} \delta_i^m \delta_k^n$$

where  $\tau_{ij} = \{x_i, x_j\}$ ,  $\sum_j \tau_{ij} \tau_{jk} = -\delta_{ik}$  and  $\Delta(x_m x_n) = \Delta(x_n x_m)$

Cartan Metric  $g(V, W) = \text{tr}(\text{ad}(V) \circ \text{ad}(W))$

$$g_{ij;kl} = \sum_{m,n,o,p} f_{ij;mp}^{(op)} f_{kl;op}^{(mn)} = 4(N+1)(\tau_{il} \tau_{kj} + \tau_{ik} \tau_{lj}) \text{ (non-degenerate)}$$

Proof

If  $g(V, \cdot) = 0$ , then  $\sum_{i,j} \tau_{li} \text{Sym}(V^{ij}) \tau_{jk} = 0$ . Since  $\tau$  invertible,  $V^{ij}$  must be antisymmetric (while  $\text{Sym}(V^{ij}) \neq 0$ )  $\Rightarrow V = 0$ .

$\Rightarrow$  Second-order moments form a **semisimple Lie algebra!**

# Cartan structure of second-order moments

## Cartan subalgebra $\mathfrak{h}$

Moments  $\Delta(q_i\pi_i)$  for  $1 \leq i \leq N$  span the Cartan subalgebra.

- $\{\Delta(q_i, \pi_i), \Delta(q_j, \pi_j)\} = 0, \forall i, j$
- $g(\Delta(q_i, \pi_i), \Delta(q_i, \pi_i)) = 0, \forall i \neq j$
- $g(\Delta(q_i, \pi_i), \Delta(q_i, \pi_i)) = g(\Delta(q_i, \pi_j), \Delta(q_i, \pi_j)), \forall i, j$
- $\Delta(q_i q_j), \Delta(\pi_i \pi_j)$ : nilpotent in 3 steps
- $\Delta(q_k \pi_l)$  with  $k \neq l$ : nilpotent in 2 steps

## Root system

Eigenvalues of  $\text{ad}(\Delta(q_i\pi_i))(\cdot)$  with  $\{\Delta(q_i, \pi_i), \Delta(x_j, x_k)\} \propto \Delta(x_j, x_k)$ :

- +2:  $\Delta(\pi_i \pi_i)$
- +1:  $\Delta(\pi_i x_k)$  with  $x_k \neq q_i, \pi_i$
- -1:  $\Delta(q_i x_k)$  with  $x_k \neq q_i, \pi_i$
- -2:  $\Delta(q_i q_i)$
- 0: others

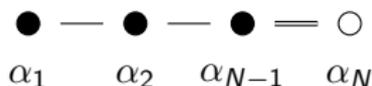
# Dynkin Diagram

## Simple Roots

$$\{\Delta(q_2\pi_1), \Delta(q_3\pi_2), \dots, \Delta(q_N\pi_{N-1}), \Delta(\pi_N^2)\}$$

Corresponding simple root vectors:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 2 \end{pmatrix}$$



**Figure:** Dynkin diagram for  $\mathfrak{sp}(2N, \mathbb{R})$  ( $C_N$ ). Filled circles: shorter roots, empty circles: longer roots.

The algebra of second-order moments is  $\mathfrak{sp}(2N, \mathbb{R})!$

# Casimir Functions

For  $\mathfrak{sp}(2N, \mathbb{R})$ , there are  $N$  Casimir functions:

$$U_{2m} \propto \text{tr}[(\tau \Delta)^{2m}], \quad m \leq N$$

where  $\Delta$  is the matrix with components  $\Delta_{ij} = \Delta(x_i x_j)$  and  $\tau_{ij} = \{x_i, x_j\}$  and

$$\{U_{2m}, H_{\text{eff},2}\} = 0, \quad \forall m$$

## Interpretation

These are **approximate constants of motion** in quantum mechanics:

- Commute with any function of second-order moments
- Do NOT necessarily commute with higher moments
- One constant per classical degree of freedom

## Example: $sp(4, \mathbb{R})$ for $N = 2$

$\{\Delta(\pi_1^2), \Delta(\pi_1 q_1), \Delta(q_1^2), \Delta(\pi_2^2), \Delta(\pi_2 q_2), \Delta(q_2^2), \Delta(\pi_1 \pi_2), \Delta(\pi_1 q_2), \Delta(\pi_2 q_1), \Delta(q_1 q_2)\}$

Cartan metric:

$$g = \begin{pmatrix} 0 & 0 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \end{pmatrix}$$

Cartan subalgebra:  $\mathfrak{h} = \langle \Delta(q_1 \pi_1), \Delta(q_2 \pi_2) \rangle$

Simple root vectors  $\{\Delta(\pi_1 q_2), \Delta(\pi_2^2)\}$  implying simple roots:

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

# Third-Order truncation ( $N = 1$ and $o = 3$ )

At third order, we have:

- 7 moments total
- Brackets between third-order moments = 0 (within truncation)
- Third-order moments form an **Abelian ideal**

## Examples for Poisson brackets

$$\{\Delta(q^2), \Delta(q^2\pi)\} = 2\Delta(q^3)$$

$$\{\Delta(q^2), \Delta(q\pi^2)\} = 4\Delta(q^2\pi)$$

$$\{\Delta(q^2), \Delta(\pi^3)\} = 6\Delta(q\pi^2)$$

- Lie algebra:  $\mathfrak{sp}(2, \mathbb{R}) \ltimes \mathbb{R}^4$
- Using the isomorphism to  $\mathfrak{sp}(2, \mathbb{R})$ :  $A = -\frac{1}{2}\Delta(\pi^2)$ ,  $B = \frac{1}{2}\Delta(q^2)$ ,  $C = \Delta(q\pi)$
- Casimir:  $K = -\frac{1}{2}(AB + BA) - \frac{1}{4}C^2 = -\frac{15}{4}\mathbb{I} = -\frac{3}{2}\left(\frac{3}{2} + 1\right)\mathbb{I}$
- This is the **spin-3/2 representation** of  $\mathfrak{sp}(2, \mathbb{R})$ !

## Definition

A *canonical realization* of an algebra  $(C^\infty(M), \{\cdot, \cdot\})$  on  $\tilde{M} \subset M$  is a homomorphism  $(C^\infty(\tilde{M}), \{\cdot, \cdot\}) \rightarrow (C^\infty(\mathbb{R}^{2r} \times \mathbb{R}^l), \{\cdot, \cdot\}_{\text{can}})$  to functions on  $\mathbb{R}^{2r} \times \mathbb{R}^l$  with:

- $\{s_n, p^m\}_{\text{can}} = \delta_n^m$  on  $\mathbb{R}^{2r}$ ,
- $\{f, U\}_{\text{can}} = 0$  for all  $f$  and  $U \in \mathbb{R}^l$ .

It is **faithful** if  $\dim M = 2r + l$  and  $2r = \text{rank}$  (of the Poisson tensor on  $M$ ).

## Example (su(2))

- Poisson Brackets:  $\{S_i, S_j\} = \sum_k \epsilon_{ijk} S_k$  and Casimir:  $S^2 = \sum_i S_i^2$
- Canonical Realization:

$$S_x = \sqrt{S^2 - S_z^2} \cos \phi \quad \text{and} \quad S_y = \sqrt{S^2 - S_z^2} \sin \phi$$

$$\text{Check: } \{\phi, S_z\} = \{\arctan(S_y/S_x), S_z\} = 1$$

# Second-Order Truncation (N=1)

Known result [Prezhdo 2006]:

## Canonical Realization

$$\Delta(q^2) = s^2$$

$$\Delta(q\pi) = sp_s$$

$$\Delta(\pi^2) = p_s^2 + \frac{U}{s^2}$$

$U$  is the Casimir function:

$$U = \Delta(q^2)\Delta(\pi^2) - \Delta(q\pi)^2 \geq \frac{\hbar^2}{4}$$

Maps to  $\mathfrak{sp}(2, \mathbb{R})$  via:  $[A, B] = C$ ,  $[A, C] = -2A$ ,  $[B, C] = 2B$

$$A = -\frac{1}{2}\Delta(\pi^2), \quad B = \frac{1}{2}\Delta(q^2), \quad C = \Delta(q\pi)$$

# Poisson structure of semiclassical truncations

## Casimir-Darboux coordinates

Find a (faithful) mapping  $\mathcal{F}$  from  $\Delta^i$  to Casimir-Darboux coordinates  $(s_k, p^k)$ , such that the Poisson bracket of Poisson manifold  $\mathcal{P}$  is preserved:

$$\mathcal{F} : \{\Delta^i\} \rightarrow (U_j, s_k, p^k), \quad 0 < i \leq \dim(\mathcal{P}), \quad 0 < j \leq \dim(\text{Null}(\mathcal{P})), \quad 0 < k \leq r.$$

with properties

$$\{s_n, p^m\} = \delta_n^m, \quad \{\cdot, U_j\} = 0,$$

$$\mathcal{P}^{ij}(\Delta) = \{\Delta^i, \Delta^j\} = \sum_{n=1}^r \frac{\partial \Delta^i}{\partial s_n} \frac{\partial \Delta^j}{\partial p^n} - \frac{\partial \Delta^i}{\partial p^n} \frac{\partial \Delta^j}{\partial s_n}$$

- The number of  $\Delta^i$  in terms  $r$  and  $o$  (semiclassical order):

$$\dim(\mathcal{P}) = \frac{1+o}{2r} \binom{2r+o}{o+1} - (2r+1)$$

# Casimir-Darboux coordinates ( $N = 1$ and $o = 2$ )

- Based on the proof of Darboux theorem [Arnold 1997]: We construct hypersurfaces Poisson-orthogonal to already-found canonical pairs
- Consider  $s = \sqrt{\Delta(q^2)}$  as first canonical coordinate.
- Identify the parameter along its Hamiltonian flow with  $-p_s$ :

$$\frac{\partial \Delta(q^2)}{\partial p_s} = -\{\Delta(q^2), s\} = 0$$

$$\frac{\partial \Delta(q\pi)}{\partial p_s} = -\{\Delta(q\pi), s\} = s$$

$$\frac{\partial \Delta(\pi^2)}{\partial p_s} = -\{\Delta(\pi^2), s\} = 2 \frac{\Delta(q\pi)}{s}$$

- Integrate:

$$\Delta(q\pi) = sp_s + f_1(s)$$

$$\Delta(\pi^2) = p_s^2 + 2 \frac{f_1(s)}{s} p_s + f_2(s)$$

# Casimir-Darboux coordinates ( $N = 1$ and $o = 2$ ) - Result

Impose  $\{\Delta(q\pi), \Delta(\pi^2)\} = 2\Delta(\pi^2)$ :

$$\begin{aligned}\frac{df_1}{ds} &= \frac{f_1}{s} \\ \frac{df_2}{ds} &= 2\frac{f_1}{s^2} \frac{df_1}{ds} - 2\frac{f_2}{s}\end{aligned}$$

Solutions:  $f_1(s) = \tilde{U}s$ ,  $f_2(s) = \frac{U}{s^2} + \tilde{U}^2$

After canonical transformation  $p_s \rightarrow p_s + \tilde{U}$ :

## Casimir-Darboux Coordinates

$$\Delta(q^2) = s^2, \quad \Delta(q\pi) = sp_s, \quad \Delta(\pi^2) = p_s^2 + \frac{U}{s^2}$$

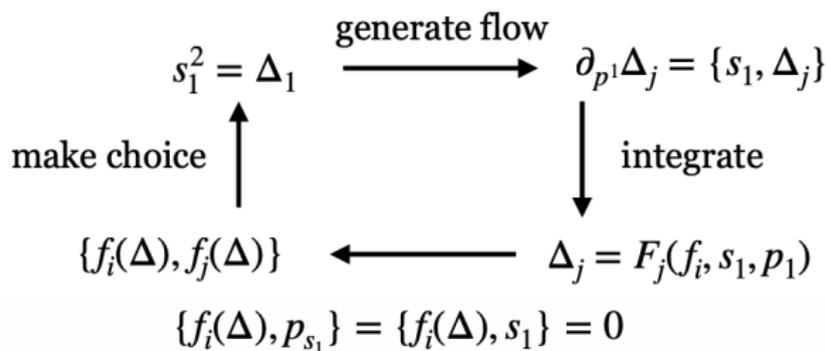
$U$  is the Casimir:

$$U = \Delta(q^2)\Delta(\pi^2) - \Delta(q\pi)^2 \geq \frac{\hbar^2}{4}$$

# General Method for $N \geq 2$ and $o = 2$

After finding  $(s_1, p_1)$ :

- 1 Construct  $\dim(\mathcal{P}) - 2$  independent functions  $f_i$  with  $\{f_i, s_1\} = 0 = \{f_i, p_1\}$
- 2 These are **Dirac observables** w.r.t.  $(s_1, p_1)$
- 3 Their Poisson brackets are closed  $\Rightarrow$  new Poisson manifold
- 4 Iterate: choose  $s_2$  from among the  $f_i$
- 5 Continue until full dimension reached or all Casimirs found



# $N = 2$ and $o = 2$ : Overview and Result-1

## Canonical Variables

- $(s_1, p_1), (s_2, p_2)$ : fluctuation variables for each degrees of freedom.
- $(\alpha, p_\alpha, \beta, p_\beta)$ : cross-correlation variables
- $C_1, C_2$ : Casimir functions

For classical pairs  $i = 1, 2$

$$\Delta(q_i^2) = s_i^2, \quad \Delta(q_i p_i) = s_i p_i, \quad \Delta(\pi_i^2) = p_i^2 + \frac{U_i(\alpha, p_\alpha, \beta, p_\beta, C_1, C_2)}{s_i^2}$$

where

$$U_1 = (p_\alpha - p_\beta)^2 + \frac{1}{2 \sin^2 \beta} \left( (C_1 - 4p_\alpha^2) - \sqrt{C_2 - C_1^2 + (C_1 - 4p_\alpha^2)^2 \sin(\alpha + \beta)} \right)$$
$$U_2 = (p_\alpha + p_\beta)^2 + \frac{1}{2 \sin^2 \beta} \left( (C_1 - 4p_\alpha^2) - \sqrt{C_2 - C_1^2 + (C_1 - 4p_\alpha^2)^2 \sin(\alpha - \beta)} \right)$$

## $N = 2$ and $o = 2$ : Cross-correlations

$$\Delta(q_1 q_2) = s_1 s_2 \cos \beta$$

$$\Delta(q_1 \pi_2) = s_1 p_2 \cos \beta - \frac{s_1}{s_2} (p_\alpha + p_\beta) \sin \beta$$

$$\Delta(q_2 \pi_1) = s_2 p_1 \cos \beta - \frac{s_2}{s_1} (p_\alpha - p_\beta) \sin \beta$$

$$\Delta(\pi_1 \pi_2) = p_1 p_2 \cos \beta - \frac{p_1}{s_2} (p_\alpha + p_\beta) \sin \beta + \frac{p_2}{s_1} (p_\alpha - p_\beta) \sin \beta$$

$$\frac{1}{s_1 s_2} (p_\alpha^2 - p_\beta^2) \cos \beta - \frac{1}{2 s_1 s_2 \sin^2 \beta} \\ \times \left( (C_1 - 4 p_\alpha^2) \cos \beta - \sqrt{C_2 - C_1^2 + (C_1 - 4 p_\alpha^2)^2 \sin \alpha} \right)$$

### Interpretations

- $\cos \beta = \frac{\Delta(q_1 q_2)}{\sqrt{\Delta(q_1^2) \Delta(q_2^2)}}$  measures correlation ( $\beta = \pi/2 \Rightarrow$  uncorrelated)
- $\alpha$  appears only in pure momentum moments  $\Rightarrow$  impurity parameter

# Effective potentials for $N = 2$ and $o = 2$

- Classical Hamiltonian and with two coupled degrees of freedom:

$$H_{\text{class}} = \frac{1}{2}(\pi_1^2 + \pi_2^2) + V(q_1, q_2)$$

- Canonically realized quantum Hamiltonian up to second semi-classical order:

$$H_Q = H_{\text{class}} + \frac{1}{2}p_{s_1}^2 + \frac{1}{2}p_{s_2}^2 + \frac{U_1(p_\alpha, p_\beta, \alpha, \beta)}{2s_1^2} + \frac{U_2(p_\alpha, p_\beta, \alpha, \beta)}{2s_2^2} \\ + \frac{1}{2}V_{11}s_1^2 + \frac{1}{2}V_{22}s_2^2 + V_{12}s_1s_2 \cos \beta + \mathcal{O}(\hbar^{3/2})$$

- The uncertainties of the two separate oscillators are coupled by the canonical angles

# Solution for low-energy effective potential

- Take “quantum” momenta (conjugate to fluctuation variables) to zero ( $p_i = 0$ ) and look for the global minimum via  $\nabla H_Q = 0$ , we find the effective potential:

$$V_{\text{eff}} = V(q_1, q_2) + \frac{\hbar}{2} \sqrt{\frac{1}{2} \left( V_{11} + V_{22} + \sqrt{(V_{11} - V_{22})^2 + 4V_{12}^2} \right)} \\ + \frac{\hbar}{2} \sqrt{\frac{1}{2} \left( V_{11} + V_{22} - \sqrt{(V_{11} - V_{22})^2 + 4V_{12}^2} \right)}$$

- This gives the correct ground state energy of two harmonic oscillators with the classical coupling of  $V_{\text{int}} = \gamma \omega^2 q_1 q_2$ :

$$E_{\text{ground}} = \frac{1}{2} \hbar \omega (\sqrt{1 + \gamma} + \sqrt{1 - \gamma})$$

- The corrections for  $V_{\text{eff}}$  can be written as:  $\frac{\hbar}{2} \sum \sqrt{\text{eig} \left( \frac{\partial^2 V}{\partial q_i \partial q_j} \right)}$ . This form easily generalizes to many degrees of freedom

# Summary

## Main Results

- Systematic method to construct faithful canonical realizations of semiclassical truncations
- Second-order truncations isomorphic to  $\mathfrak{sp}(2N, \mathbb{R})$
- Complete faithful realization for 2 degrees of freedom at second order (4 canonical pairs + 2 Casimirs)
- Third-order truncation:  $\mathfrak{sp}(2, \mathbb{R}) \ltimes \mathbb{R}^4$  (spin-3/2 representation)

## Key Results

- Second order ( $N = 1$ ): known result
- **New:** Multiple degrees of freedom and higher orders
- Connection to  $\mathfrak{sp}(2N, \mathbb{R})$  Lie algebras
- Faithful realizations (correct number of degrees of freedom)

## Applications

- Parametric resonance in coupled oscillators
- Tunneling in atom ionization
- Effective potentials
- Quantum cosmology

## Future Directions

- Canonical (faithful) realization of  $\mathfrak{sp}(2N, \mathbb{R})$  (one-loop QFT)
- Higher-order truncations for multiple degrees of freedom (already have some preliminary results)
- Applications to interacting field theories
- Further quantum cosmology applications