The Good, the Bad, and the Degenerate

Constraint Structures of Nonlinear PDEs

A Dirac–Bergmann Perspective

Ilmar Gahramanov

Boğaziçi University

February 27, 2026

Istanbul Integrability and Stringy Topics Initiative

https://istringy.org

- **IStringy** is a research group in theoretical and mathematical physics.
- Based at the Department of Physics, Boğaziçi University.
- Focused on:
  - string theory,
  - supersymmetric field theories,
  - integrable models.
- Strong involvement of **undergraduate students**. Early exposure to:
  - research papers,
  - seminar culture,
  - solid foundation of theoretical and mathematical physics.

The Constraint Cartel

Ali Pazarcı

Nadir Ghazanfari

This talk Is Based On

- *Hamiltonian Formalism for Nonlinear Schrödinger Equations*

  Ali Pazarcı, Umut Can Turhan, Nader Ghazanfari, Ilmar Gahramanov, **Commun. Nonlinear Sci. Numer. Simul. 121 (2023) 107191**, **arXiv:2205.14721**

- Ongoing/Upcoming work with A.Pazarcı and N.Ghazanfari

Motivation

- Many integrable PDEs admit Lagrangian formulations.
- However, these Lagrangians are often **degenerate**.
- Standard Legendre transform fails.
- Need Dirac–Bergmann constraint formalism.

Even the Universe Has Constraints

The same constraint structure appears in:

- Mini-superspace quantum cosmology
  - models with scalar fields
- Modified gravity
  - Higher-derivative cosmological actions

In these models:

- Dirac–Bergmann identifies all constraints.
- Physical phase space is constructed correctly.

Why the Algorithm is Essential

Dirac-Bergmann:

- Classifies constraints:
  - First-class $\rightarrow$ Gauge symmetry
  - Second-class $\rightarrow$ Physical reduction
- Constructs Dirac brackets.
- Produces consistent canonical quantization.

Without proper constraint analysis, quantization may introduce spurious degrees of freedom.

The same structure appears in integrable PDEs.

When the Hessian Says No

Degeneracy condition:

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0$$

- Canonical momenta cannot be inverted.
- Constraints must be introduced.

Dirac-Bergmann Algorithm

The Four-Step Survival Guide

Step 1: Define momenta

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Step 2: Identify primary constraints

Step 3: Define total Hamiltonian

$$H_T = H_C + \lambda_a \phi_a$$

Step 4: Impose consistency

$$\dot{\phi}_a = \{\phi_a, H_T\} \approx 0$$

May produce secondary constraints.

A Lagrangian density $\mathcal{L}[\psi_i, \dot{\psi}_i]$ is **degenerate** if its Hessian vanishes:

$$\det\left(\frac{\delta^2 \mathcal{L}}{\delta\dot{\psi}_i \delta\dot{\psi}_j}\right) = 0.$$

Primary constraints arise from canonical momenta:

$$\pi_{\psi_i} = \frac{\delta \mathcal{L}}{\delta \dot{\psi}_i}.$$

The total Hamiltonian is

$$H = \int dx \left(\mathcal{H}_{\text{can}} + \sum_i \lambda_i \, c_i\right),$$

where

$$\mathcal{H}_{\text{can}} = \sum_i \pi_{\psi_i} \dot{\psi}_i - \mathcal{L}.$$

Consistency $\{c_i, H\} \approx 0$ may fix Lagrange multipliers λ_i or yield *secondary constraints*.

KdV Equation

$$u_t - 6u\, u_x + u_{xxx} = 0$$

This simple-looking equation quietly reshaped mathematical physics.

Water waves. Solitons. Integrability.

And yes - its Lagrangian is degenerate.

Introduce auxiliary fields:

$$u = \phi_x, \qquad \psi = \phi_{xx}.$$

Lagrangian density:

$$\mathcal{L}_{\mathsf{KdV}} = \tfrac{1}{2}\,\phi_t\,\phi_x + \phi_x^3 + \phi_x\,\psi_x + \tfrac{1}{2}\,\psi_x^2.$$

[Nutku, 1984], [Nutku, 2000]

KdV Primary Constraints

Canonical momenta:

$$c_1 = \pi_\psi, \qquad c_2 = \pi_\phi - \tfrac{1}{2}\,\phi_x.$$

Total Hamiltonian:

$$H = -\phi_x^3 - \phi_x\psi_x - \tfrac{1}{2}\,\psi^2 + \lambda_1 c_1 + \lambda_2 c_2.$$

Wait ... There's More!

Consistency condition:

$$\{c_1, H\} = \psi - \phi_{xx}.$$

This must vanish, giving the secondary constraint

$$\tilde{c}_3 = \psi - \phi_{xx}.$$

The Hamiltonian is enlarged by adding a multiplier for $\tilde{c}_3$.

The total Hamiltonian including constraints:

$$H_{\mathrm{KdV}} = -\phi_x^3 - \phi_x \psi_x - \frac{1}{2}\psi^2 + \lambda_1 c_1 + \lambda_2 c_2 + \tilde{\lambda}_3 \tilde{c}_3.$$

Could the secondary constraint be a consequence of the higher-order derivative structure of the field u?

Total Hamiltonian Structure

After imposing consistency conditions and fixing multipliers:

$$
\begin{aligned}
H = &\frac{1}{2}\phi_x^3 + \frac{1}{2}\psi^2 + \phi_x\psi_x + \frac{1}{2}\phi_{xx}^2 \\
&- \pi_\phi\left(\phi_{3x} + 3\phi_x^2\right) \\
&- \pi_\psi\left(\phi_{5x} + 6\phi_{xx}^2 + 6\phi_x\phi_{3x}\right).
\end{aligned}
$$

Eventually, the equations of motion are obtained as follows:

$$\phi_t + 3\phi_x^2 + \phi_{3x} = 0,$$

$$\phi_{xt} + 6\phi_x\phi_{xx} + \phi_{4x} = 0,$$

$$\psi_t + 6\phi_{xx}^2 + 6\phi_x\phi_{3x} + \phi_{5x} = 0,$$

$$\psi - \phi_{xx} = 0.$$

One equation. Infinite structure.

From the constraint dynamics of the KdV equation, the construction of a Hamiltonian from a degenerate Lagrangian with secondary constraints can be followed explicitly.

Interestingly, KdV admits infinitely many Lagrangian formulations and therefore infinitely many symplectic structures.

[Nutku, 2000] [Nutku, Pavlov 2002]

A possible alternative construction of the KdV Lagrangian uses the Schwarzian derivative.

ongoing work with Y.Yıldırım, A.Kahraman and Ş.Çetin

Cubic Nonlinear Schrödinger Equation

$$iu_t + u_{xx} + 2|u|^2 u = 0.$$

Set

$$u = \phi e^{i\theta}.$$

Real evolution equations:

$$\phi_t = -\theta_{xx}\phi - 2\phi_x\theta_x,$$

$$\phi\,\theta_t = 2\phi^3 + \phi_{xx} - \phi\,\theta_x^2.$$

Cubic NLSE Lagrangian

$$\mathcal{L}_{\text{nls}} = -\tfrac{1}{2}\,\theta_t\,\phi^2 + \tfrac{1}{2}\,\phi^4 - \tfrac{1}{2}\,\phi_x^2 - \tfrac{1}{2}\,\theta_x^2\phi^2.$$

The Hessian determinant vanishes, so the system is degenerate.

Cubic NLSE Constraints

Canonical momenta:

$$\pi_\phi = 0, \qquad \pi_\theta = -\tfrac{1}{2}\,\phi^2.$$

Primary constraints:

$$c_1 = \pi_\phi, \qquad c_2 = \pi_\theta + \tfrac{1}{2}\,\phi^2.$$

Total Hamiltonian density:

$$\mathcal{H}_{\mathrm{nls}} = -\tfrac{1}{2}\,\phi^4 + \tfrac{1}{2}\,\phi_x^2 + \tfrac{1}{2}\,\theta_x^2\phi^2 + \lambda_1\pi_\phi + \lambda_2\Big(\pi_\theta + \tfrac{1}{2}\phi^2\Big).$$

Cubic NLSE Final Hamiltonian

After substitution:

$$\mathcal{H}_{\text{nls}} = \tfrac{1}{2}\,\phi^4 + \pi_\phi(-\phi\theta_{xx} - 2\phi_x\theta_x) + \pi_\theta\left(2\phi^2 - \theta_x^2 + \frac{\phi_{xx}}{\phi}\right).$$

Hamilton equations reproduce the real evolution equations.
If we substitute the canonical momenta into the total Hamiltonian and perform an integration by parts, we obtain

$$H_{\text{nls}} = \int dx\left(\frac{1}{2}\phi_x^2 + \frac{1}{2}\phi^2\theta_x^2 - \frac{1}{2}\phi^4\right).$$

This is a redefined version of

$$H_{\text{nls}} = \int dx\left(\frac{1}{2}|\psi_x|^2 - \frac{1}{2}|\psi|^4\right), \qquad \psi = \phi e^{i\theta}.$$

The original integrability influencers: [Zakharov, Shabat 1972]

Same Constraints, Different Personality

Logarithmic Nonlinear Schrödinger Equation

$$iu_t + u_{xx} + u \ln |u|^2 = 0.$$

Lagrangian:

$$\mathcal{L}_{\mathrm{lnls}} = -\tfrac{1}{2}\,\phi^2\theta_t - \tfrac{1}{2}\,\phi_x^2 - \tfrac{1}{2}\,\phi^2\theta_x^2 + \phi^2 \ln \phi - \tfrac{1}{2}\,\phi^2.$$

Primary constraints are identical to the cubic case. The total Hamiltonian density takes the form

$$H_{\mathrm{lnls}} = \frac{1}{2}\phi^2 + \pi_\phi\big(-\phi\theta_{xx} - 2\phi_x\theta_x\big) + \pi_\theta \left(\frac{\phi_{xx}}{\phi} - \theta_x^2 + 2\ln\phi \right).$$

Hamilton equations derived from this Hamiltonian reproduce the logarithmic nonlinear Schrödinger equation.

Future Directions

- **The Next Level: Supersymmetry**
  - Construction of a consistent Lagrangian formulation for supersymmetric extensions of NLSE.
  - Analysis of degeneracy structure in the presence of fermionic degrees of freedom.
- **Classification of Degenerate Lagrangians**
  - Systematic classification of degenerate Lagrangians based on their constraint hierarchy.
  - Distinction between:
    - Primary-only constraint systems
    - Systems with secondary constraints
  - Relation between dispersion order and constraint structure.

Toward a structural classification of nonlinear evolution equations.

The Moral of the Story

- Degenerate Lagrangians arise naturally in nonlinear evolution equations.
- The Dirac-Bergmann algorithm provides a systematic procedure to construct a consistent Hamiltonian formulation.
- For second-order nonlinear Schrödinger equations (cubic and logarithmic), only primary constraints appear.
- The specific form of the nonlinearity does not alter the constraint structure.
- Higher-order dispersion terms (such as in KdV or fourth-order NLSE) may lead to secondary constraints.

The 2nd Gülbahçe School and Workshop
for Theoretical Physics

Teşekkürler!